W-ALGEBRAS AND CHIRAL DIFFERENTIAL OPERATORS AT THE CRITICAL LEVEL

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ABSTRACT. Let $\mathcal{A}_{crit}$ be the chiral algebra corresponding to the affine Kac-Moody algebra at the critical level $\hat{\mathfrak{g}}_{crit}$. Let $\mathfrak{Z}_{crit}$ be the center of $\mathcal{A}_{crit}$. The commutative chiral algebra $\mathfrak{Z}_{crit}$ admits a canonical deformation into a non-commutative chiral algebra $\mathcal{W}_\hbar$. In this paper we will express the resulting first order deformation via the chiral algebra $\mathcal{D}_{crit}$ of chiral differential operators on $G((t))$ at the critical level.

1. INTRODUCTION

1.1. Let X be a smooth curve over the complex numbers with structure sheaf $\mathcal{O}_X$ and sheaf of differential operators $\mathcal{D}_X$. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let κ be an invariant non-degenerate bilinear form

$$\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}.$$

Note that the condition on $\mathfrak{g}$ being simple implies that κ is a multiple of the Killing form κ_{kill}. Let $\hat{\mathfrak{g}}_\kappa$ be the affine Kac-Moody algebra given as the central extension of the loop algebra $\mathfrak{g}((t))$

$$(1) \qquad\qquad 0 \to \mathbb{C}\mathbb{1} \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0,$$

with bracket given by

$$[af(t), bg(t)] = [a,b]f(t)g(t) + \kappa(a,b)\mathrm{Res}(fdg) \cdot \mathbb{1},$$

where a and b are elements in $\mathfrak{g}$, and $\mathbb{1}$ is the central element.

Our basic tool in this paper is the theory of chiral algebras. In particular we will use the chiral algebra $\mathcal{A}_\kappa$ attached to $\hat{\mathfrak{g}}_\kappa$ as defined in [AG]. We will assume the reader is familiar with the foundational work [BD] on this subject. However we will briefly recall some basic definitions and notations. Throughout this paper $\Delta : X \hookrightarrow X \times X$ will denote the diagonal embedding and $j : U \to X \times X$ its complement, where $U = (X \times X) - \Delta(X)$.

For any two sheaves $\mathcal{M}$ and $\mathcal{N}$ denote by $\mathcal{M} \boxtimes \mathcal{N}$ the external tensor product $\pi_1^*\mathcal{M} \underset{\mathcal{O}_{X \times X}}{\otimes} \pi_2^*\mathcal{N}$, where π_1 and π_2 are the two projections from $X \times X$ to X.

For a right $\mathcal{D}_X$-module $\mathcal{M}$ define the extension $\Delta_!(\mathcal{M})$ as

$$\Delta_!(\mathcal{M}) = j_*j^*(\Omega_X \boxtimes \mathcal{M})/\Omega_X \boxtimes \mathcal{M}.$$

Sections of $\Delta_!(\mathcal{M})$ can be thought as distributions on $X \times X$ with support on the diagonal and with values on $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{N}$ are two right $\mathcal{D}_X$-modules, we will denote by $\mathcal{M} \overset{!}{\otimes} \mathcal{N}$ the right $\mathcal{D}_X$-module $\mathcal{M} \otimes \mathcal{N} \otimes \Omega_X^*$.

Recall that a *chiral algebra* over X is a right $\mathcal{D}_X$-module $\mathcal{A}$ endowed with a *chiral bracket*, i.e. with a map of $\mathcal{D}_{X^2}$-modules

$$\mu : j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_!(\mathcal{A})$$

which is antisymmetric and satisfies the Jacobi identity.

We will denote by $[\,,\,]_{\mathcal{A}}$ the restriction of μ to $\mathcal{A} \boxtimes \mathcal{A} \hookrightarrow j_* j^*(\mathcal{A} \boxtimes \mathcal{A})$.

By a *commutative chiral algebra* we mean a chiral algebra $\mathcal{R}$ such that $[\,,\,]_{\mathcal{R}}$ vanishes. In other words it is a chiral algebra such that the chiral bracket μ factors as

$$j_* j^*(\mathcal{R} \boxtimes \mathcal{R}) \to \Delta_!(\mathcal{R} \overset{!}{\otimes} \mathcal{R}) \to \Delta_!(\mathcal{R}).$$

Equivalently, $\mathcal{R}$ can be described as a right $\mathcal{D}_X$-module with a commutative product on the corresponding left $\mathcal{D}_X$-module $\mathcal{R}^l := \mathcal{R} \otimes \Omega_X^*$. For instance, in the case $\mathcal{R} = \Omega_X$, with chiral product defined as $\mu(f(x,y)dx \boxtimes dy) = f(x,y)dx \wedge dy \pmod{\Omega_{X \times X}^2}$, you simply recover the commutative product on the sheaf of functions on X, which is in fact the left $\mathcal{D}_X$-module corresponding to Ω_X.

Consider now the chiral algebra $\mathcal{A}_\kappa$ attached to $\hat{\mathfrak{g}}_\kappa$. For $\kappa = \kappa_{crit} = -\frac{1}{2}\kappa_{kill}$ denote by $\mathfrak{Z}_{crit}$ the center of $\mathcal{A}_{crit} := \mathcal{A}_{\kappa_{crit}}$. This is a commutative chiral algebra with the property that the fiber $(\mathfrak{Z}_{crit})_x$ over any point $x \in X$ is equal to the commutative algebra $\mathrm{End}_{\hat{\mathfrak{g}}_{crit}}(\mathbb{V}_{\mathfrak{g},crit}^0)$, where

$$\mathbb{V}_{\mathfrak{g},crit}^0 := \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}}^{\hat{\mathfrak{g}}_{crit}} \mathbb{C}.$$

Note that in the above definition we are using the fact that the exact sequence (1) splits over $\mathfrak{g}[[t]] \subset \mathfrak{g}((t))$, hence we can regard the direct sum $\mathfrak{g}[[t]] \oplus \mathbb{C}$ as a subalgebra of $\hat{\mathfrak{g}}_{crit}$.

The chiral algebra $\mathfrak{Z}_{crit}$ is closely related to the center of the twisted enveloping algebra of $\hat{\mathfrak{g}}_{crit}$. For any chiral algebra $\mathcal{A}$ and any point $x \in X$, we can form an associative topological algebra $\hat{\mathcal{A}}_x$ with the property that its discrete continuous modules are the same as $\mathcal{A}$-modules supported at x (see [BD] 3.6.6). In this case the topological associative algebra corresponding to $\mathfrak{Z}_{crit}$ is isomorphic to the center of the appropriately completed twisted enveloping algebra $U'(\hat{\mathfrak{g}}_{crit})$ of $\hat{\mathfrak{g}}_{crit}$, where $U'(\hat{\mathfrak{g}}_{crit})$ denotes $U(\hat{\mathfrak{g}}_{crit})/(\mathbb{1}-1)$, (here 1 denotes the identity element in $U(\hat{\mathfrak{g}}_{crit})$).

The importance of choosing the level κ to be κ_{crit} relies on the fact that the center $\hat{\mathfrak{Z}}_{crit}$ of $U'(\hat{\mathfrak{g}}_{crit})$ happens to be very big, unlike any other level $\kappa \neq \kappa_{crit}$ where the center is in fact just $\mathbb{C}$, as shown in [FF]. Another crucial feature of the critical level is that $\hat{\mathfrak{Z}}_{crit}$ carries a natural Poisson structure,

obtained by considering the one parameter deformation of κ_{crit} given by $\kappa_{crit} + \hbar\kappa_{kill}$, as explained below in the language of chiral algebras. Moreover, according to [FF, F1, F2], the center $\hat{\mathfrak{Z}}_{crit}$ is isomorphic, as Poisson algebra, to the *quantum Drinfeld-Sokolov reduction* of $U'(\hat{\mathfrak{g}}_{crit})$ introduced in [FF]. In particular the above reduction provides a quantization of the Poisson algebra $\hat{\mathfrak{Z}}_{crit}$ that will be central in this article.

Since the language we have chosen is the one of chiral algebras, we will now reformulate these properties for the algebra $\mathfrak{Z}_{crit}$.

The commutative chiral algebra $\mathfrak{Z}_{crit}$ can be equipped with a Poisson structure which can be described in either of the following two equivalent ways:

- For any $\hbar \neq 0$ let κ be any non critical level $\kappa = \kappa_{crit} + \hbar\kappa_{kill}$ and denote by $\mathcal{A}_\hbar$ the chiral algebra $\mathcal{A}_\kappa$. Let z and w be elements of $\mathfrak{Z}_{crit}$. Let z_κ and w_κ be any two families of elements in $\mathcal{A}_\hbar$ such that $z = z_\kappa$ and $w = w_\kappa$ when $\hbar = 0$. Define the Poisson bracket of z and w to be

$$\{z, w\} = \frac{[z_\kappa, w_\kappa]_{\mathcal{A}_\hbar}}{\hbar} \pmod{\hbar}.$$

- The functor Ψ of semi-infinite cohomology introduced in [FF] (which is the analogous of the quantum Drinfeld-Sokolov reduction mentioned before and whose main properties will be recalled later), produces a 1-parameter family of chiral algebras $\{\mathcal{W}_\hbar\} := \{\Psi(\mathcal{A}_\hbar)\}$ such that $\mathcal{W}_0 \simeq \mathfrak{Z}_{crit}$. Define the Poisson structure on $\mathfrak{Z}_{crit}$ as

$$\{z, w\} = \frac{[\widetilde{z}_\hbar, \widetilde{w}_\hbar]_{\mathcal{W}_\hbar}}{\hbar} \pmod{\hbar}$$

  where $z = \widetilde{z}_\hbar|_{\hbar=0}$ and $w = \widetilde{w}_\hbar|_{\hbar=0}$.

Although the above two expressions look the same, we'd like to stress the fact that, unlike the second construction, in the first we are not given any deformation of $\mathfrak{Z}_{crit}$. In other words the elements z_κ and w_κ do not belong to the center of $\mathcal{A}_\kappa$ (that in fact is trivial). It is worth noticing that the associative topological algebras $\hat{\mathcal{W}}_h$ associated to them (usually denoted by $W_\hbar$) are the well known *W-algebras*.

As in the case of usual algebras, the Poisson structure on $\mathfrak{Z}_{crit}$ gives the sheaf of Kähler differentials $\Omega^1(\mathfrak{Z}_{crit})$ a structure of Lie* algebroid. A remarkable feature, when dealing with chiral algebras, is that the existence of a quantization $\{\mathcal{W}_\hbar\}$ of $\mathfrak{Z}_{crit}$ allows us to construct what is called a *chiral extension* $\Omega^c(\mathfrak{Z}_{crit})$ of the Lie* algebroid $\Omega^1(\mathfrak{Z}_{crit})$, and moreover, as it is explained in [BD] 3.9.11, this establishes an equivalence of categories between 1-st order quantizations of $\mathfrak{Z}_{crit}$ and chiral extensions of $\Omega^1(\mathfrak{Z}_{crit})$. This equivalence is the point of departure for this paper.

We will now recall the definitions of the main objects we will be using (a more detailed description can be found in [BD] Section 3.9.).

Since we will only deal with right $\mathcal{D}_X$-modules, for any two right $\mathcal{D}_X-$modules $\mathcal{M}$ and $\mathcal{N}$, we will simply write $\mathcal{M} \otimes \mathcal{N}$ instead of $\mathcal{M} \overset{!}{\otimes} \mathcal{N}$. We will also assume that our modules are flat as $\mathcal{O}_X$-modules. Therefore we always have an exact sequence

$$0 \to \mathcal{M} \boxtimes \mathcal{N} \to j_* j^* (\mathcal{M} \boxtimes \mathcal{N}) \to \Delta_! (\mathcal{M} \otimes \mathcal{N}) \to 0,$$

(note that we are already adopting the different notation for the tensor product).

Let $(\mathcal{R}, m : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R})$ be a commutative chiral algebra.

Definition 1.1. Let L be a Lie* algebra acting by derivations on $\mathcal{R}$ via a map τ. An $\mathcal{R}$-*extension* of L is a $\mathcal{D}_X$-module L^c fitting in the short exact sequence

$$0 \to \mathcal{R} \to L^c \xrightarrow{\pi} L \to 0$$

together with a Lie* algebra structure on L^c such that π is a morphism of Lie* algebras and the adjoint action of L^c on $\mathcal{R} \subset L^c$ coincides with $\tau \circ \pi$.

Definition 1.2. A Lie* $\mathcal{R}$-algebroid $\mathcal{L}$ is a Lie* algebra with a central action of $\mathcal{R}$ (a map $\mathcal{R} \otimes \mathcal{L} \to \mathcal{L}$) and a Lie* action $\tau_{\mathcal{L}}$ of $\mathcal{L}$ on $\mathcal{R}$ by derivations such that

- $\tau_{\mathcal{L}}$ is $\mathcal{R}$-linear with respect to the $\mathcal{L}$-variable.

- The adjoint action of $\mathcal{L}$ is a $\tau_{\mathcal{L}}$-action of $\mathcal{L}$ (as a Lie* algebra) on $\mathcal{L}$ (as an $\mathcal{R}$-module).

In the next definitions we consider objects equipped with a chiral action of $\mathcal{R}$ instead of just a central one.

Definition 1.3. Let $\mathcal{R}$ be a commutative chiral algebra, and $\mathcal{L}$ be a Lie* $\mathcal{R}$-algebroid. A *chiral $\mathcal{R}$-extension* of $\mathcal{L}$ is a $\mathcal{D}_X$-module $\mathcal{L}^c$ such that

$$(2) \qquad\qquad 0 \to \mathcal{R} \xrightarrow{i} \mathcal{L}^c \to \mathcal{L} \to 0,$$

together with a Lie* bracket and a chiral $\mathcal{R}$-module structure $\mu_{\mathcal{R},\mathcal{L}^c}$ on $\mathcal{L}^c$ satisfying the following properties:

- The arrows in (2) are compatible with the Lie* algebra and chiral $\mathcal{R}$-module structures.

- The chiral operations $\mu_{\mathcal{R}}$ and $\mu_{\mathcal{R},\mathcal{L}^c}$ are compatible with the Lie* actions of $\mathcal{L}^c$.

- The * operation that corresponds to $\mu_{\mathcal{R},\mathcal{L}^c}$ (i.e. the restriction of $\mu_{\mathcal{R},\mathcal{L}^c}$ to $\mathcal{R} \boxtimes \mathcal{L}^c$) is equal to $-i \circ \sigma \circ \tau_{\mathcal{L}^c,\mathcal{R}} \circ \sigma$, where $\tau_{\mathcal{L}^c,\mathcal{R}}$ is the $\mathcal{L}^c$-action on $\mathcal{R}$ given by the projection $\mathcal{L}^c \to \mathcal{L}$ and the $\mathcal{L}$ action

$\tau_{\mathcal{L},\mathcal{R}}$ on $\mathcal{R}$ and σ is the transposition of variables. In other words the following diagram commutes

$$\mathcal{R} \boxtimes \mathcal{L}^c \longrightarrow j_* j^* (\mathcal{R} \boxtimes \mathcal{L}^c) \xrightarrow{\mu_{\mathcal{R},\mathcal{L}^c}} \Delta_!(\mathcal{L}^c).$$

$$\mathcal{R} \boxtimes \mathcal{L} \xrightarrow{\sigma \circ \tau_{\mathcal{L},\mathcal{R}} \circ \sigma} \Delta_!(\mathcal{R}) \quad \xrightarrow{\Delta_!(i)}$$

The previous definition can be extended by replacing $\mathcal{R}$ with any chiral algebra $\mathcal{C}$ endowed with a central action of $\mathcal{R}$. More precisely a *chiral* $\mathcal{C}$-*extension* of $\mathcal{L}$ is a $\mathcal{D}_X$-module $\mathcal{L}^c$ such that

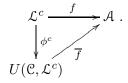$$(3) \qquad\qquad 0 \to \mathcal{C} \to \mathcal{L}^c \to \mathcal{L} \to 0,$$

together with a Lie* bracket and a chiral $\mathcal{R}$-module structure $\mu_{\mathcal{R},\mathcal{L}^c}$ on $\mathcal{L}^c$ such that:

- The arrows in (3) are compatible with the Lie* algebra and chiral $\mathcal{R}$-module structures.

- The chiral operations $\mu_{\mathcal{C}}$ and $\mu_{\mathcal{R},\mathcal{L}^c}$ are compatible with the Lie* actions of $\mathcal{L}^c$.

- The structure morphism $\mathcal{R} \to \mathcal{C}$ is compatible with the Lie* actions of $\mathcal{L}^c$.

- The * operation that corresponds to $\mu_{\mathcal{R},\mathcal{L}^c}$ (i.e. $\mu_{\mathcal{R},\mathcal{L}^c}$ restricted to $\mathcal{R} \boxtimes \mathcal{L}^c$) is equal to $-i \circ \sigma \circ \tau_{\mathcal{L}^c,\mathcal{R}} \circ \sigma$, where $\tau_{\mathcal{L}^c,\mathcal{R}}$ is the $\mathcal{L}^c$-action on $\mathcal{R}$, σ is the transposition of variables and i is the composition of the structure morphism $\mathcal{R} \to \mathcal{C}$ and the embedding $\mathcal{C} \subset \mathcal{L}^c$.

Definition 1.4. The *chiral envelope* of the chiral extension $(\mathcal{R}, \mathcal{C}, \mathcal{L}^c, \mathcal{L})$ is a pair $(U(\mathcal{C}, \mathcal{L}^c), \phi^c)$, where $U(\mathcal{C}, \mathcal{L}^c)$ is a chiral algbera and ϕ^c is a homomorphism of $\mathcal{L}^c$ into $U(\mathcal{C}, \mathcal{L}^c)$, satisfying the following universal property. For every chiral algebra $\mathcal{A}$ and any morphism $f : \mathcal{L}^c \to \mathcal{A}$ such that:

- f is a morphism of Lie* algebras.

- f restricts to a morphism of chiral algebras on $\mathcal{C} \subset \mathcal{L}^c$.

- f is a morphism of $\mathcal{R}$-modules (where the $\mathcal{R}$-action on $\mathcal{A}$ is the one given by the above point),

there exist a unique map $\overline{f} : U(\mathcal{C}, \mathcal{L}^c) \to \mathcal{A}$ that makes the following diagram commutative

$$\mathcal{L}^c \xrightarrow{f} \mathcal{A} \ .$$
$$\downarrow{\phi^c} \qquad \nearrow{\overline{f}}$$
$$U(\mathcal{C}, \mathcal{L}^c)$$

It is shown in [BD] that such object exists. When $\mathcal{C} = \mathcal{R}$ we will simply write $U(\mathcal{L}^c)$ instead of $U(\mathcal{R}, \mathcal{L}^c)$.

1.2. CHIRAL EXTENSIONS OF $\Omega^1(\mathcal{R})$. Let $\mathcal{R}$ be a commutative chiral algebra equipped with a Poisson bracket $\{\,,\,\} : \mathcal{R} \boxtimes \mathcal{R} \to \Delta_!(\mathcal{R})$. We will refer to such object as *chiral-Poisson algebra*. As we mentioned before, any Poisson structure on a commutative chiral algebra $\mathcal{R}$ gives the sheaf $\Omega^1(\mathcal{R})$ a structure of a Lie* algebroid. Now consider the following: to a chiral extension

$$0 \to \mathcal{R} \to \Omega^c(\mathcal{R}) \to \Omega^1(\mathcal{R}) \to 0,$$

consider the pull-back of the above sequence via the differential $d : \mathcal{R} \to \Omega^1(\mathcal{R})$. The resulting short exact sequence is a $\mathbb{C}[\hbar]/\hbar^2$-deformation of the chiral-Poisson algebra $\mathcal{R}^1$. If we denote by $\mathcal{Q}^{ch}(\mathcal{R})$ the groupoid of $\mathbb{C}[\hbar]/\hbar^2$-deformations of the chiral-Poisson algebra $\mathcal{R}$, and by $\mathcal{P}^{ch}(\Omega^1(\mathcal{R}))$ the groupoid of chiral $\mathcal{R}$-extensions of $\Omega^1(\mathcal{R})$, the above map defines a functor

$$\mathcal{P}^{ch}(\Omega^1(\mathcal{R})) \to \mathcal{Q}^{ch}(\mathcal{R}).$$

In [BD] 3.9.10. the following is shown.

Theorem 1.1. *The above functor defines an equivalence between* $\mathcal{P}^{ch}(\Omega^1(\mathcal{R}))$ *and* $\mathcal{Q}^{ch}(\mathcal{R})$.

Consider now the following diagram:

$$
\begin{array}{ccc}
\mathcal{P}^{ch}(\Omega^1(\mathcal{R})) & \longrightarrow & \{\text{Lie}^* \text{ algebroid structures on } \Omega^1(\mathcal{R})\} \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
\mathcal{Q}^{ch}(\mathcal{R}) & \longrightarrow & \{\text{Chiral-Poisson structures on } \mathcal{R}\}.
\end{array}
$$

A natural question to ask is the following: if we consider the quantization of $\mathfrak{Z}_{crit}$ given by applying the functor of semi-infinite cohomology to $\mathcal{A}_\hbar$, how does the corresponding chiral extension of $\Omega^1(\mathfrak{Z}_{crit})$ look like? The answer to the above question is the main body of this paper.

We will give an explicit construction of $\Omega^c(\mathcal{R})$ for an arbitrary chiral-Poisson algebra $\mathcal{R}$. In the case where $\mathcal{R} = \mathfrak{Z}_{crit}$ we will see how this chiral extension relates to the *chiral algebra of differential operators on the loop group* $G((t))$ at the critical level introduced in [AG], where G is the algebraic group of adjoint type corresponding to $\mathfrak{g}$.

1.3. MAIN THEOREM. Recall from [AG] the definition of the chiral algebra of twisted differential operators on $G((t))$. Denote such algebra by $\mathcal{D}_\hbar$ (note that this notation differs from the one in [AG] where the same object was

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[1]If $\{\,,\,\}$ denotes the Poisson bracket on $\mathcal{R}$, this is indeed a quantization of $(\mathcal{R}, 2\{\,,\,\})$

denoted by $\mathcal{D}_\kappa$, with $\kappa = \kappa_{crit} + \hbar\kappa_{kill}$). As it is explained there, the fiber $(\mathcal{D}_\hbar)_x$ of $\mathcal{D}_\hbar$ at $x \in X$ is isomorphic to

$$(\mathcal{D}_\hbar)_x \simeq U(\hat{\mathfrak{g}}_\kappa) \underset{U(\mathfrak{g}[[t]] \oplus \mathbb{C})}{\otimes} \mathcal{O}_{G[[t]]}.$$

Moreover $\mathcal{D}_\hbar$ comes equipped with two embeddings

(4) $$\mathcal{A}_\hbar \xrightarrow{l_\hbar} \mathcal{D}_\hbar \xleftarrow{r_\hbar} \mathcal{A}_{-\hbar}$$

corresponding to left and right invariant vector fields on the loop group $G((t))$. In particular, for $\hbar = 0$, we have $\mathcal{A}_\hbar = \mathcal{A}_{-\hbar} = \mathcal{A}_{crit}$ and $\mathcal{D}_\hbar = \mathcal{D}_{crit}$. Therefore we obtain two different embeddings, $l := l_0$ and $r := r_0$ of $\mathcal{A}_{crit}$ into $\mathcal{D}_{crit}$

$$\mathcal{A}_{crit} \xrightarrow{l} \mathcal{D}_{crit} \xleftarrow{r} \mathcal{A}_{crit}.$$

If we restrict these two embeddings to $\mathfrak{Z}_{crit}$, as it is explained in [FG] Theorem 5.4, we have

$$l(\mathfrak{Z}_{crit}) = l(\mathcal{A}_{crit}) \cap r(\mathcal{A}_{crit}) = r(\mathfrak{Z}_{crit}).$$

Moreover the two compositions

$$\mathfrak{Z}_{crit} \hookrightarrow \mathcal{A}_{crit} \xrightarrow{l} \mathcal{D}_{crit} \xleftarrow{r} \mathcal{A}_{crit} \hookleftarrow \mathfrak{Z}_{crit}$$

are intertwined by the automorphism $\eta : \mathfrak{Z}_{crit} \to \mathfrak{Z}_{crit}$ given by the involution of the Dynkin diagram that sends a weight λ to $-w_0(\lambda)$ (i.e. when restricted to $\mathfrak{Z}_{crit}$ we have $l = r \circ \eta$).

1.4. The two embedding l and r of $\mathcal{A}_{crit}$ into $\mathcal{D}_{crit}$ endow the fiber $(\mathcal{D}_{crit})_x$ with a structure of $\hat{\mathfrak{g}}_{crit}$-bimodule. The fiber can therefore be decomposed according to these actions as explained below.

Denote by $\hat{\mathfrak{Z}}_{crit}$ the center of the completion of the enveloping algebra of $\hat{\mathfrak{g}}_{crit}$. For a dominant weight λ, let V^λ be the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight λ and let $\mathbb{V}^\lambda_{\mathfrak{g},crit}$ be the $\hat{\mathfrak{g}}_{crit}$-module given by

$$\mathbb{V}^\lambda_{\mathfrak{g},crit} := U(\hat{\mathfrak{g}}_{crit}) \underset{U(\mathfrak{g}[[t]] \oplus \mathbb{C})}{\otimes} V^\lambda.$$

The action of the center $\hat{\mathfrak{Z}}_{crit}$ on $\mathbb{V}^\lambda_{\mathfrak{g},crit}$ factors as follows

$$\hat{\mathfrak{Z}}_{crit} \twoheadrightarrow \mathfrak{z}^\lambda_{crit} := \mathrm{End}(\mathbb{V}^\lambda_{\mathfrak{g},crit}).$$

Denote by I^λ the kernel of the above map, and consider the formal neighborhood of $\mathrm{Spec}(\mathfrak{z}^\lambda_{crit})$ inside $\mathrm{Spec}(\hat{\mathfrak{Z}}_{crit})$. Let $\hat{\mathfrak{g}}_{crit}$-mod$^{G[[t]]}$ be the full subcategory of $\hat{\mathfrak{g}}_{crit}$-modules such that the action of $\mathfrak{g}[[t]]$ can be integrated to an action of $G[[t]]$. We have the following Lemma.

Lemma 1.2. *Any module M in $\hat{\mathfrak{g}}_{crit}$-mod$^{G[[t]]}$ can be decomposed into a direct sum of submodules M_λ such that each M_λ admits a filtration whose subquotients are annihilated by I^λ.*

As a bimodule over $\hat{\mathfrak{g}}_{crit}$ the fiber at any point $x \in X$ of $\mathcal{D}_{crit}$ is $G[[t]]$ integrable with respect to both actions, hence we have two direct sum decompositions of $(\mathcal{D}_{crit})_x$ corresponding to the left and right action of $\hat{\mathfrak{g}}_{crit}$. These decompositions coincide up to the involution η and we have

$$(\mathcal{D}_{crit})_x = \bigoplus_{\lambda\, dominant} (\mathcal{D}_{crit})_x^\lambda,$$

where $(\mathcal{D}_{crit})_x^\lambda$ is the direct summand supported on the formal completion of $\mathrm{Spec}(\mathfrak{z}_{crit}^\lambda)$.

Denote by $\mathcal{D}_{crit}^0$ the $\mathcal{D}_X$-module corresponding to $(\mathcal{D}_{crit})_x^0$. It is easy to see that $\mathcal{D}_{crit}^0$ is in fact a chiral algebra.

Since the fiber of $\mathcal{A}_{crit}$ at x is isomorphic to $\mathbb{V}_{\mathfrak{g},crit}^0$, the embeddings l and r must land in the chiral algebra $\mathcal{D}_{crit}^0$. Hence we have

$$\mathcal{A}_{crit} \xrightarrow{l,r} \mathcal{D}_{crit}^0 \longhookrightarrow \mathcal{D}_{crit}.$$

The above two embeddings give $\mathcal{D}_{crit}^0$ a structure of $\mathcal{A}_{crit}$-bimodule, hence it makes sense to apply the functor of semiinfinite cohomology Ψ to it twice (as it will explained in 2.3). Let us denote by $\mathcal{B}^0$ the resulting chiral algebra

$$\mathcal{B}^0 := (\Psi \boxtimes \Psi)(\mathcal{D}_{crit}^0).$$

The main result of this paper is the following.

Theorem 1. The chiral envelope $U(\Omega^c(\mathfrak{Z}_{crit}))$ of the extension

$$0 \to \mathfrak{Z}_{crit} \to \Omega^c(\mathfrak{Z}_{crit}) \to \Omega(\mathfrak{Z}_{crit}) \to 0,$$

given by the quantization $\{\mathcal{W}_\hbar := \Psi(\mathcal{A}_\hbar)\}$ of the center $\mathfrak{Z}_{crit}$, is isomorphic to the chiral algebra $\mathcal{B}^0$.

1.5. STRUCTURE OF THE PROOF. The proof of Theorem 1 will be organized as follows: in Section 2 we will give an alternative formulation of the Theorem that consists in finding a map F from $\Omega^c(\mathfrak{Z}_{crit})$ to $\mathcal{B}^0$ with some particular properties. The definition of the above map will rely on the explicit construction of the chiral extension $\Omega^c(\mathfrak{Z}_{crit})$ that will be given in Section 3. In Section 4 we will finally define the map F and conclude the proof of the Theorem.

2. Reformulation of the Theorem

2.1. In this section we will show how to prove the Theorem 1 assuming the existence of a map F from $\Omega^c(\mathfrak{Z}_{crit})$ to $\mathcal{B}^0$. In order to do so, we will use the fact that both $U(\Omega^c(\mathfrak{Z}_{crit}))$ and $\mathcal{B}^0$ can be equipped with filtrations as explained below.

2.2. The chiral algebra $U(\Omega^c(\mathfrak{Z}_{crit}))$, being the chiral envelope of the extension

$$0 \to \mathfrak{Z}_{crit} \to \Omega^c(\mathfrak{Z}_{crit}) \to \Omega^1(\mathfrak{Z}_{crit}) \to 0,$$

has its standard *Poincaré-Birkhoff-Witt filtration*. In fact, more generally, given a chiral-extension $(\mathcal{R}, \mathcal{C}, \mathcal{L}^c, \mathcal{L})$, using the notations from Definition 1.4, we can define a PBW filtration on $U(\mathcal{C}, \mathcal{L}^c)$ as the filtration generated by $U(\mathcal{C}, \mathcal{L}^c)_0 := \phi^c(\mathcal{C})$ and

$$U(\mathcal{C}, \mathcal{L}^c)_1 := \mathrm{Im}(j_* j^*(\mathcal{L}^c \boxtimes \mathcal{C}) \xrightarrow{\phi^c \boxtimes \phi^c|_e}$$
$$\to j_* j^*(U(\mathcal{C}, \mathcal{L}^c) \boxtimes U(\mathcal{C}, \mathcal{L}^c)) \to \Delta_!(U(\mathcal{C}, \mathcal{L}^c))).$$

Moreover in [BD] 3.9.11. the following theorem is proved.

Theorem 2.1. *If $\mathcal{R}$ and $\mathcal{C}$ are $\mathcal{O}_X$ flat and $\mathcal{L}$ is a flat $\mathcal{R}$-module then we have an isomorphism*

$$\mathcal{C} \underset{\mathcal{R}}{\otimes} Sym_{\mathcal{R}}^{\cdot} \mathcal{L} \xrightarrow{\cong} gr_{\cdot} U(\mathcal{C}, \mathcal{L}^c).$$

By applying the above to the case where $\mathcal{C} = \mathcal{R} = \mathfrak{Z}_{crit}$ and the extension of $\mathcal{L} = \Omega^1(\mathfrak{Z}_{crit})$ given by $\mathcal{L}^c = \Omega^c(\mathfrak{Z}_{crit})$ we get

$$gr_{\cdot} U(\Omega^c(\mathfrak{Z}_{crit})) \simeq \mathrm{Sym}_{\mathfrak{Z}_{crit}}^{\cdot} \Omega^1(\mathfrak{Z}_{crit}).$$

2.3. The filtration on $\mathcal{B}^0$ is defined using the functor Ψ of *semi-infinite cohomology* introduced in [FF].

Recall that, for any central charge $\kappa = \hbar \kappa_{kill} + \kappa_{crit}$, the functor Ψ assigns to a chiral $\mathcal{A}_\hbar$-module a $\Psi(\mathcal{A}_\hbar) = \mathcal{W}_\hbar$-module. In particular, for every chiral algebra $\mathcal{B}$, and every morphism of chiral algebras $\phi : \mathcal{A}_\hbar \to \mathcal{B}$ we have

$$\Psi : \left\{ \begin{array}{c} \text{chiral algebra morphism} \\ \phi : \mathcal{A}_\hbar \to \mathcal{B} \end{array} \right\} \to \left\{ \begin{array}{c} \text{chiral algebra morphism} \\ \Psi(\phi) : \mathcal{W}_\hbar \to \Psi(\mathcal{B}) \end{array} \right\}.$$

Moreover recall that for $\hbar = 0$ we have $\Psi(\mathcal{A}_{crit}) \simeq \mathfrak{Z}_{crit}$.

As it is explained in [FG], the chiral algebra $\mathcal{B}^0$ can be described as

$$(\Psi \boxtimes \Psi)(U(\mathcal{C}, \mathcal{L}^c)) \xrightarrow{\sim} \mathcal{B}^0 = (\Psi \boxtimes \Psi)(\mathcal{D}_{crit}^0),$$

for some particular chiral algebra $\mathcal{C}$ and chiral extension $\mathcal{L}^c$. Hence it carries a canonical filtration induced by the PBW-filtration on $U(\mathcal{C}, \mathcal{L}^c)$. We will recall below the definitions of these algebras.

THE RENORMALIZED CHIRAL ALGEBROID. Recall that [FG] Proposition 4.5. shows the existence of a chiral extension $\mathcal{A}^{ren,\tau}$ that fits into the following exact sequence

$$0 \to (\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \to \mathcal{A}^{ren,\tau} \to \Omega^1(\mathfrak{Z}_{crit}) \to 0,$$

which is a chiral extension of $(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit})$ in the sense we introduced in Definition 1.3. In particular, if we consider the chiral envelope $U((\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})$, by Theorem 2.1 we have

$$\mathrm{gr}.(U((\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})) \simeq$$

$$\simeq (\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \underset{\mathfrak{Z}_{crit}}{\otimes} \mathrm{Sym}^{\cdot}_{\mathfrak{Z}_{crit}}(\Omega^1(\mathfrak{Z}_{crit})).$$

The chiral envelope $U((\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})$ is closely related to the chiral algebra $\mathcal{D}^0_{crit}$, in fact in [FG] the following is proved:

Theorem 2.2. *We have an embedding G of the chiral extension $\mathcal{A}^{ren,\tau}$ into $\mathcal{D}_{crit}$ such that the maps l and r are the compositions of this embedding with the canonical maps*

$$\mathcal{A}_{crit} \rightrightarrows (\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \xrightarrow{G} \mathcal{A}^{ren,\tau}.$$

The embedding extends to a homomorphism of chiral algebras

$$U((\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau}) \to \mathcal{D}_{crit}$$

and the latter is an isomorphism into $\mathcal{D}^0_{crit}$.

Therefore we see that $\mathcal{B}^0$ is given by applying the functor $\Psi \boxtimes \Psi$ to the chiral envelope $U(\mathcal{C}, \mathcal{L}^c)$, for

$$\mathcal{C} = (\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \text{ and } \mathcal{L}^c = \mathcal{A}^{ren,\tau}.$$

In particular, since the functor Ψ is exact, we obtain a filtration on $\mathcal{B}^0$ induced from the PBW-filtration on $U(\mathcal{A}_{crit} \otimes \mathcal{A}_{crit}, \mathcal{A}^{ren,\tau})$ such that

$$\mathrm{Sym}^{\cdot}_{\mathfrak{Z}_{crit}}\Omega^1(\mathfrak{Z}_{crit}) \xrightarrow{\sim} \mathrm{gr}.\mathcal{B}^0,$$

where we used the fact that $\Psi(\mathcal{A}_{crit}) \simeq \mathfrak{Z}_{crit}$.

2.4.   Note that if we apply the functor Ψ to the two embeddings in (4), we obtain two embeddings

$$\mathcal{W}_{\hbar} \xrightarrow{l_{\hbar}} (\Psi \boxtimes \Psi)(\mathcal{D}_{\hbar}) \xleftarrow{r_{\hbar}} \mathcal{W}_{-\hbar}$$

such that $l := l_0 = r_0 \circ \eta =: r \circ \eta$, where we are denoting simply by $l_{\hbar}$ and $r_{\hbar}$ the maps $\Psi(l_{\hbar})$ and $\Psi(r_{\hbar})$ respectively. In particular, for $\hbar = 0$, we obtain two embeddings l and r of $\mathfrak{Z}_{crit}$ into $(\Psi \boxtimes \Psi)(\mathcal{D}_{crit})$ that differs by η.

Moreover the image of the two maps lands in $\mathcal{B}^0$, therefore we obtain two embeddings

$$\mathfrak{Z}_{crit} \xrightarrow{l} \mathcal{B}^0 \xleftarrow{r} \mathfrak{Z}_{crit}.$$

From the above construction it is clear that $\mathfrak{Z}_{crit}$ corresponds to the 0-th part of the filtration defined on $\mathcal{B}^0$. Moreover, by the definition of the map G from Theorem 2.2 (see [FG]), the embedding $\mathfrak{Z}_{crit} \hookrightarrow \mathcal{B}^0$ induced by the inclusion $(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \hookrightarrow U((\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})$ under $\Psi \boxtimes \Psi$, coincides with l.

2.5. Suppose now that we are given a map $F : \Omega^c(\mathfrak{Z}_{crit}) \to \mathcal{B}^0$ satisfying the conditions stated in Definition 1.4. By the universal property of the chiral envelope, we automatically get a map

$$U(\Omega^c(\mathfrak{Z}_{crit})) \to \mathcal{B}^0.$$

Clearly not every such map will induce an isomorphism between the two chiral algebras. Theorem 1 can be reformulated as saying that there exists a map as above, that gives rise to an isomorphism $U(\Omega^c(\mathfrak{Z}_{crit})) \xrightarrow{\sim} \mathcal{B}^0$. More precisely we have the following:

Theorem 2. There exists a map $F : \Omega^c(\mathfrak{Z}_{crit}) \to (\mathcal{B}^0)_1 \hookrightarrow \mathcal{B}^0$ compatible with the $\mathfrak{Z}_{crit}$ structure on both sides that restricts to the embedding l of chiral algebras on $\mathfrak{Z}_{crit}$ such that the following diagram commutes

We will now show how Theorem 1 follows from Theorem 2. The proof Theorem 2 will occupy the rest of the article.

Proof of (Theorem 2 $\Rightarrow$ Theorem 1). To prove Theorem 1 we need to show that the above F induces an isomorphism $U(\Omega^c(\mathfrak{Z}_{crit})) \xrightarrow{\sim} \mathcal{B}^0$. This amounts to showing that the following diagram commutes for every i:

$$(5) \qquad \mathrm{gr}_{i+1} U(\Omega^c(\mathfrak{Z}_{crit})) \xrightarrow{\quad F \quad} \mathrm{gr}_{i+1} \mathcal{B}^0.$$

But this follows from the fact that the above filtrations are generated by their first two terms. In fact, more generally, for any chiral envelope $U(\mathcal{L}^c)$, we have

$$\Delta_!(\mathrm{gr}_{i+1} U(\mathcal{L}^c)) :=$$

$$= \mathrm{Im} \left(\begin{array}{c} j_* j^* (U(\mathcal{L}^c)_1 \boxtimes U(\mathcal{L}^c)_i) \to \\ \Delta_!(U(\mathcal{L}^c)) \end{array} \right) \bigg/ \mathrm{Im} \left(\begin{array}{c} j_* j^* (U(\mathcal{L}^c)_1 \boxtimes U(\mathcal{L}^c)_{i-1}) \to \\ \Delta_!(U(\mathcal{L}^c)) \end{array} \right).$$

It is not hard to see that the isomorphism $\mathrm{Sym}^{i+1}_{\mathfrak{Z}_{crit}} \Omega^1(\mathfrak{Z}_{crit}) \xrightarrow{\sim} \mathrm{gr}_{i+1} U(\Omega^c(\mathfrak{Z}_{crit}))$ (and similarly for $\mathcal{B}^0$) is the one induced by the map

$$j_* j^* (\Omega^1(\mathfrak{Z}_{crit}) \boxtimes \mathrm{Sym}^i_{\mathfrak{Z}_{crit}}) \to \Delta_!(\mathrm{gr}_{i+1} U(\Omega^c(\mathfrak{Z}_{crit}))),$$

that in fact vanishes when restricted to $\Omega^1(\mathfrak{Z}_{crit}) \boxtimes \mathrm{Sym}^i_{\mathfrak{Z}_{crit}}$, and factors through the action of $\mathfrak{Z}_{crit}$. Therefore the diagram (5) commutes by induction on i.

$\square$

3. CONSTRUCTION OF THE CHIRAL EXTENSION $\Omega^c(\mathfrak{Z}_{crit})$

3.1.   As we saw in the previous section, the proof of Theorem 2 amounts to the construction of a particular map F from $\Omega^c(\mathfrak{Z}_{crit})$ to $\mathcal{B}^0$. However the construction of such map requires a more explicit description of $\Omega^c(\mathfrak{Z}_{crit})$. In fact recall that, for every commutative-Poisson chiral algebra $\mathcal{R}$ and quantization $\{\mathcal{R}_h\}$ of the Poisson structure, there is canonically associated a chiral extension

$$0 \to \mathcal{R} \to \Omega^c(\mathcal{R}) \to \Omega^1(\mathcal{R}) \to 0$$

given by the equivalence of category stated in Theorem 1.1. However the proof of this theorem doesn't provide a construction of it. This section will be devoted to the construction of the above extension.

3.2.   Starting from the Lie* algebra extension

$$0 \to \mathcal{R} \to \mathcal{R}^c \to \mathcal{R} \to 0,$$

where $\mathcal{R}^c := \mathcal{R}_\hbar / \hbar^2 \mathcal{R}_\hbar$ acts on $\mathcal{R}$ via the projection $\mathcal{R}^c \to \mathcal{R}$ and the Poisson bracket on $\mathcal{R}$, we will first construct a chiral extension (see Definition 1.3) $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$ fitting into

$$0 \to \mathcal{R} \to \mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c) \to \mathcal{R} \otimes \mathcal{R} \to 0,$$

where $\mathcal{R} \otimes \mathcal{R}$ is viewed as a Lie* algebroid using the Poisson structure on $\mathcal{R}$. The chiral extension $\Omega^c(\mathcal{R})$ will be then defined as a quotient $\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)$. More generally, in 3.3-3.10 we will explain how to construct a chiral extension $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ fitting into

(6) $$0 \to \mathcal{R} \to \mathrm{Ind}^{ch}_{\mathcal{R}}(L^c) \to \mathcal{R} \otimes L \to 0$$

for every Lie* algebra L acting on $\mathcal{R}$ by derivations and every extension

$$0 \to \mathcal{R} \to L^c \to L \to 0.$$

The case where $L = \mathcal{R}$ and $L^c = \mathcal{R}^c$, will be presented in 3.11 as a particular case of the above general construction.

3.3. DEFINITION OF $\mathrm{IND}^{ch}_{\mathcal{R}}(L^c)$. Let $(\mathcal{R}, \mu)$ be a commutative chiral algebra and let L be a Lie* algebra acting on $\mathcal{R}$ by derivations via the map τ. The induced $\mathcal{R}$-module $\mathcal{R} \otimes L$ has a unique structure of Lie* $\mathcal{R}$-algebroid such that the morphism $1_{\mathcal{R}} \otimes id_L : L \to \mathcal{R} \otimes L$ is a morphism of Lie* algebras compatible with their actions on $\mathcal{R}$. Note that we have an obvious map

$$i : L \to \mathcal{R} \otimes L.$$

The Lie* algebroid $\mathcal{R} \otimes L$ is called *rigidified*. More generally we have the following definition.

Definition 3.1. A Lie* algebroid $\mathcal{L}$ is called *rigidified* if we are given a Lie* algebra L acting on $\mathcal{R}$ via the map τ, and an inclusion $i : L \to \mathcal{L}$, such that $\mathcal{R} \otimes L \xrightarrow{\sim} \mathcal{L}$.

3.4. Let $\mathcal{L}$ be a rigidified Lie* algebroid. Consider the map that sends a chiral extension of $\mathcal{L}$

$$0 \to \mathcal{R} \to \mathcal{L}^c \to \mathcal{L} \to 0$$

to the $\mathcal{R}$ extension of L given by considering the pull-back of the map $i : L \to \mathcal{L}$.

Denote by $\mathcal{P}^{cl}(\mathcal{L})$ (resp. $\mathcal{P}^{ch}(\mathcal{L})$) the groupoid of classical (resp. chiral) extensions of $\mathcal{L}$ (where by classical we mean extensions in the category of Lie* algebroids), and by $\mathcal{P}(L, \tau)$ the Picard groupoid of $\mathcal{R}$-extensions of L. Clearly the map mentioned above (that can be equally defined for classical extensions as well), defines two functors

$$\mathcal{P}^{cl}(\mathcal{L}) \to \mathcal{P}(L, \tau), \quad \mathcal{P}^{ch}(\mathcal{L}) \to \mathcal{P}(L, \tau).$$

As it is explained in [BD] 3.9.9. the following is true.

Proposition 3.1. *If L is $\mathcal{O}_X$ flat, then these maps define an equivalence of groupoids*

$$(7) \qquad \mathcal{P}^{cl}(\mathcal{L}) \xrightarrow{\sim} \mathcal{P}(L, \tau), \quad \mathcal{P}^{ch}(\mathcal{L}) \xrightarrow{\sim} \mathcal{P}(L, \tau),$$

Given a Lie* algebra extension $0 \to \mathcal{R} \to L^c \to L \to 0$, define $\mathrm{Ind}^{cl}_{\mathcal{R}}(L^c)$ (resp. $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$) to be the classical (resp. chiral) extension corresponding to the above sequence under the equivalences stated in the above proposition.

3.5. In subsections 3.6-3.7 we will briefly recall the construction of the inverse functors to (7) in the classical and chiral setting respectively (as presented in [BD]). However in 3.10 we will give a different construction of the inverse functor in the chiral setting, i.e. a different construction of the chiral extension $\mathrm{Ind}^{ch}_{\mathcal{R}}(L^c)$ associated to any $\mathcal{R}$-extension of L. The latter construction will be used to define the chiral extension $\Omega^c(\mathcal{R})$.

3.6. For the "classical" map $\mathcal{P}^{cl}(\mathcal{L}) \to \mathcal{P}(L, \tau)$, to an extension

(8) $$0 \to \mathcal{R} \to L^c \to L \to 0,$$

the inverse functor associates the classical extension $\mathrm{Ind}_{\mathcal{R}}^{cl}(L^c)$ of the Lie* algebroid $\mathcal{R} \otimes L = \mathcal{L}$ given by the push-out of the extension

$$0 \to \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes L^c \to \mathcal{R} \otimes L \to 0$$

via the map $m : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}$.

3.7. The construction of the inverse functor in the "chiral" setting given in [BD] (i.e. the construction of $\mathrm{Ind}_{\mathcal{R}}^{ch}(L^c)$), uses the following two facts:

- $\mathcal{P}^{ch}(\mathcal{L})$ has a structure of $\mathcal{P}^{cl}(\mathcal{L})$-torsor under Baer sum.

- $\mathcal{P}^{ch}(\mathcal{L})$ is non empty.

The first fact follows from condition 3) in the definition of chiral $\mathcal{R}$-extension, which guarantees that the Baer difference of two chiral extensions is a classical one. In other words the action of $\mathcal{R}$ on the sum of two chiral extensions is automatically central.

The non emptiness of $\mathcal{P}^{ch}(\mathcal{L})$ follows from the existence of a distinguished chiral $\mathcal{R}$-extension $\mathrm{Ind}_{\mathcal{R}}(L)$ attached to every Lie* algebra L acting on $\mathcal{R}$. Such object is defined by the following:

Definition-Proposition 3.2. *Suppose that we are given a Lie* algebra L acting by derivations on $\mathcal{R}$ via the map τ, and let $\mathcal{L}$ be a rigidified Lie* algebroid (see Definition 3.1), so we have a morphism of Lie* algebras $i : L \to \mathcal{L}$ such that $\mathcal{R} \otimes L \xrightarrow{\sim} \mathcal{L}$. Then there exist a chiral extension $\mathrm{Ind}_{\mathcal{R}}(L)$ equipped with a lifting $\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$ such that $\bar{i}$ is a morphism of Lie* algebras and the adjoint action of L on $\mathcal{R}$ via $\bar{i}$ equals τ. The pair $(\mathrm{Ind}_{\mathcal{R}}(L), \bar{i})$ is unique.*

The proof of this proposition can be found in [BD] 3.9.8. However in 3.8 we will recall the construction of $\mathrm{Ind}_{\mathcal{R}}(L)$ and of the map $\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$.

To finish the construction of $\mathrm{Ind}_{\mathcal{R}}^{ch}(L^c)$ (or in other words, the construction of the inverse to the functor $\mathcal{P}^{ch}(\mathcal{R}) \to \mathcal{P}(L, \tau)$), we use the classical extension $\mathrm{Ind}_{\mathcal{R}}^{cl}(L^c)$ given in 3.6 together with the $\mathcal{P}^{cl}(\mathcal{L})$-action on $\mathcal{P}^{ch}(\mathcal{L})$. To the extension $0 \to \mathcal{R} \to L^c \to L \to 0$ we associate the chiral $\mathcal{R}$-extension

$$\mathrm{Ind}_{\mathcal{R}}^{ch}(L^c) := \mathrm{Ind}_{\mathcal{R}}^{cl}(L^c) \underset{Baer}{+} \mathrm{Ind}_{\mathcal{R}}(L)$$

of $\mathcal{R} \otimes L$ by $\mathcal{R}$, where $\mathrm{Ind}_{\mathcal{R}}(L)$ is the distinguished classical extension defined in 3.2. Note that, after pulling back the extension

$$0 \to \mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}^{ch}(L^c) \to \mathcal{L} \simeq \mathcal{R} \otimes L \to 0$$

via the map $L \to \mathcal{L} \simeq \mathcal{R} \otimes L$, we obtain the Baer sum of the trivial extension (corresponding to $\mathrm{Ind}_{\mathcal{R}}(L)$) with L^c, i.e. we recover the initial Lie* extension $0 \to \mathcal{R} \to L^c \to L \to 0$ as we should.

3.8.   In this subsection we want to recall the construction and the main properties of the distinguished chiral extension $\mathrm{Ind}_{\mathcal{R}}(L)$ given by Definition-Proposition 3.2.

Given a Lie* algebra L acting on $\mathcal{R}$ by derivations, we can consider the action map $\mathcal{R} \boxtimes L \to \Delta_!(\mathcal{R})$ and consider the following push out:
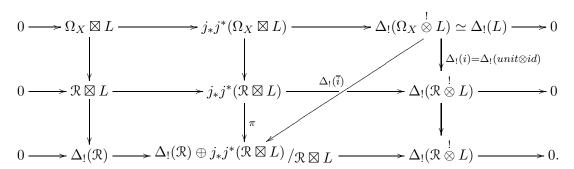
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R} \boxtimes L & \longrightarrow & j_* j^*(\mathcal{R} \boxtimes L) & \longrightarrow & \Delta_!(\mathcal{R} \otimes L) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta_!(\mathcal{R}) & \longrightarrow & \Delta_!(\mathcal{R}) \oplus j_* j^*(\mathcal{R} \boxtimes L)/\mathcal{R} \boxtimes L & \longrightarrow & \Delta_!(\mathcal{R} \otimes L) & \longrightarrow & 0.
\end{array}
$$

The term in the middle is a $\mathcal{D}_X$-module supported on the diagonal, hence by Kashiwara's Theorem (see [K] Theorem 4.30) it corresponds to a $\mathcal{D}_X$ module on X. This $\mathcal{D}_X$-module has a structure of chiral extension and will be our desired $\mathrm{Ind}_{\mathcal{R}}(L)$ (i.e. we have $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L)) \simeq \Delta_!(\mathcal{R}) \oplus j_* j^*(\mathcal{R} \boxtimes L)/\mathcal{R} \boxtimes L$).

Remark 3.3. By construction we have inclusions $\mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}(L)$ and a lifting $\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$ of $i : L \to \mathcal{R} \otimes L$. In fact we can consider the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_X \boxtimes L & \longrightarrow & j_* j^*(\Omega_X \boxtimes L) & \longrightarrow & \Delta_!(\Omega_X \overset{!}{\otimes} L) \simeq \Delta_!(L) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Delta_!(i)=\Delta_!(unit \otimes id)} & & \\
0 & \longrightarrow & \mathcal{R} \boxtimes L & \longrightarrow & j_* j^*(\mathcal{R} \boxtimes L) & \overset{\Delta_!(\bar{i})}{\longrightarrow} & \Delta_!(\mathcal{R} \overset{!}{\otimes} L) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
0 & \longrightarrow & \Delta_!(\mathcal{R}) & \longrightarrow & \Delta_!(\mathcal{R}) \oplus j_* j^*(\mathcal{R} \boxtimes L)/\mathcal{R} \boxtimes L & \longrightarrow & \Delta_!(\mathcal{R} \overset{!}{\otimes} L) & \longrightarrow & 0.
\end{array}
$$

By looking at the composition of the two vertical arrows in the middle, it is not hard to see that this composition factors through $\Delta_!(L)$. In fact the most left vertical arrow from $\Omega_X \boxtimes L$ to $\Delta_!(\mathcal{R})$ is zero. We define $\bar{i}$ to be the map corresponding (under the Kashiwara's equivalence) to $\Delta_!(\bar{i})$.

As it is shown in [BD] 3.3.6. the inclusions $\mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}(L)$, $\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$ and the chiral operation $j_* j^*(\mathcal{R} \boxtimes L) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}(L))$, uniquely determine a chiral action of $\mathcal{R}$ on $\mathrm{Ind}_{\mathcal{R}}(L)$ and a Lie* bracket on it. In other words they give $\mathrm{Ind}_{\mathcal{R}}(L)$ a structure of chiral $\mathcal{R}$-extension.

Note that this chiral $\mathcal{R}$-extension corresponds, under the equivalence given by Theorem 3.1 (i.e. after we pull-back the extension via the map $\psi : L \to \mathcal{R} \otimes \mathcal{L}$), to the trivial extension of L by $\mathcal{R}$ in $\mathcal{P}(L, \tau)$.

3.9.   To summarize we have seen that:

- If a Lie* algebra L acts on $\mathcal{R}$ we can construct the distinguished chiral extension $\mathrm{Ind}_{\mathcal{R}}(L)$ of $\mathcal{L}$ with a lifting $\bar{i} : L \to \mathrm{Ind}_{\mathcal{R}}(L)$ of the canonical map $i : L \to \mathcal{L}$.

- From an extension $0 \to \mathcal{R} \to L^c \to L \to 0$ we can construct a chiral extension $\mathrm{Ind}_{\mathcal{R}}^{ch}(L^c)$ with a map $L^c \to \mathrm{Ind}_{\mathcal{R}}^{ch}(L^c)$ given by the pull-back of $L \to \mathcal{L}$.

Remark 3.4. Clearly, if we have the extension $0 \to \mathcal{R} \to L^c \to L \to 0$, we can also consider L^c as a Lie* algebra acting on $\mathcal{R}$ via the projection $L^c \to \mathcal{R}$. In other words we forget about the extension and we only remember the Lie* algebra L^c. From point one of the above summary we can construct the distinguished chiral extension $\mathrm{Ind}_{\mathcal{R}}(L^c)$ corresponding to this L^c action on $\mathcal{R}$, together with a map $\bar{i} : L^c \to \mathrm{Ind}_{\mathcal{R}}(L^c)$.

3.10.   DIFFERENT CONSTRUCTION OF $\mathrm{IND}_{\mathcal{R}}^{ch}(L^c)$. We will now explain a different construction of the chiral extension

$$0 \to \mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}^{ch}(L^c) \to \mathcal{L} \simeq \mathcal{R} \otimes L \to 0$$

that will be used later to construct $\Omega^c(\mathcal{R})$.

As it is explained in the Remark 3.4, given an $\mathcal{R}-$extension

$$0 \to \mathcal{R} \xrightarrow{k} L^c \to L \to 0,$$

we can consider the action of L^c on $\mathcal{R}$ given by the projection $L^c \to L$ and construct the distinguished chiral extension $\mathrm{Ind}_{\mathcal{R}}(L^c)$. This is a chiral $\mathcal{R}$-extension fitting into

$$0 \to \Delta_!(\mathcal{R}) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)) \to \Delta_!(\mathcal{R} \otimes L^c) \to 0,$$

where $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)) \simeq {}^{\Delta_!(\mathcal{R}) \oplus j_* j^*(\mathcal{R} \boxtimes L^c)} / {}_{\mathcal{R} \boxtimes L^c}$. Since we ultimately want an extension of $\mathcal{R}$ by $\mathcal{R} \otimes L$, we have to quotient the above sequence by some additional relations. We will in fact obtain $\mathrm{Ind}_{\mathcal{R}}^{ch}(L^c)$ by taking the quotient of $\mathrm{Ind}_{\mathcal{R}}(L^c)$ by the image of the difference of two maps from $\mathcal{R} \otimes \mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}(L^c)$.

The above maps are given (under the Kashiwars's equivalence) by the following two maps from $\Delta_!(\mathcal{R} \otimes \mathcal{R})$ to $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c))$.
1) The first map is given by the composition

$$\Delta_!(\mathcal{R} \otimes \mathcal{R}) \xrightarrow{m} \Delta_!(\mathcal{R}) \longleftrightarrow \Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)).$$

2) For the second map, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R} \boxtimes \mathcal{R} & \longrightarrow & j_*j^*(\mathcal{R} \boxtimes \mathcal{R}) & \longrightarrow & \Delta_!(\mathcal{R} \otimes \mathcal{R}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle id \boxtimes k} & & \downarrow{\scriptstyle k \boxtimes id} & & \downarrow{\scriptstyle id \otimes k} & & \\
0 & \longrightarrow & \mathcal{R} \boxtimes L^c & \longrightarrow & j_*j^*(\mathcal{R} \boxtimes L^c) & \longrightarrow & \Delta_!(\mathcal{R} \otimes L^c) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
0 & \longrightarrow & \Delta_!(\mathcal{R}) & \longrightarrow & \Delta_!(\mathcal{R}) \oplus j_*j^*(\mathcal{R} \boxtimes L^c)/\mathcal{R} \boxtimes L^c & \longrightarrow & \Delta_!(\mathcal{R} \otimes L^c) & \longrightarrow & 0. \\
& & & & \underset{\Delta_!(\widetilde{\mathrm{Ind}}_{\mathcal{R}}(L^c))}{\parallel} & & & &
\end{array}$$

We claim that the composition of the two vertical arrows in the middle (i.e. $\pi \circ (k \boxtimes id)$) factors through $\Delta_!(\mathcal{R} \otimes \mathcal{R})$. In fact since the action of L^c on $\mathcal{R}$ is given by the projection $L^c \to \mathcal{R}$, the copy of $\mathcal{R}$ inside L^c via k acts by zero. Hence the composition of the left most vertical arrows is zero, which shows that there is a well defined map

$$\overline{k} : \Delta_!(\mathcal{R} \otimes \mathcal{R}) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)).$$

The quotient of $\mathrm{Ind}_{\mathcal{R}}(L^c)$ by the image of the difference of the above maps is exactly $\mathrm{Ind}_{\mathcal{R}}^{ch}(L^c)$.

Remark 3.5. Note that the inclusion $L^c \to \mathrm{Ind}_{\mathcal{R}}^{ch}(L^c)$ mentioned in the summary 3.9 corresponds to the composition

$$(9) \qquad \Delta_!(L^c) \xrightarrow{\Delta_!(\overline{i})} \Delta_!(\mathrm{Ind}_{\mathcal{R}}(L^c)) \twoheadrightarrow \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(L^c)).$$

3.11. A SPECIAL CASE: DEFORMATIONS OF $\mathcal{R}$. Now let $(\mathcal{R},\, m : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R})$ be a commutative chiral algebra given as $\mathcal{R} := \mathcal{R}_\hbar / \hbar \mathcal{R}_\hbar$, where $\{\mathcal{R}_\hbar\}$ is a family of chiral algebras. Denote by $\{\,,\,\}$ the Poisson bracket on $\mathcal{R}$ defined as

$$\{z, w\} = \frac{1}{\hbar}[z_\hbar, w_\hbar]_\hbar \quad (\mathrm{mod}\ \hbar),$$

where $z_\hbar = z \ (\mathrm{mod}\ \hbar)$, $w_\hbar = w \ (\mathrm{mod}\ \hbar)$ and $[\,,\,]_\hbar$ denotes the Lie* bracket on $\mathcal{R}_\hbar$ induced by the chiral product $\mu_\hbar$ restricted to $\mathcal{R}_\hbar \boxtimes \mathcal{R}_\hbar$.
Consider the quotient $\mathcal{R}^c = \mathcal{R}_\hbar / \hbar^2 \mathcal{R}_\hbar$. This is a Lie* algebra with bracket $[\,,\,]_c$ defined by

$$[\overline{z_\hbar}, \overline{w_\hbar}]_c = \overline{\frac{1}{\hbar}[z_\hbar, w_\hbar]_\hbar}.$$

Consider the short exact sequence

$$(10) \qquad 0 \to \mathcal{R} \xrightarrow{\cdot \hbar} \mathcal{R}^c \to \mathcal{R} \to 0,$$

and let us regard $\mathcal{R}^c$ as a Lie* algebra acting on $\mathcal{R}$ via the projection $\mathcal{R}^c \to \mathcal{R}$ followed by the Poisson bracket multiplied by[2] $1/2$. This sequence is an $\mathcal{R}$-extension of $\mathcal{R}$ in the sense we introduced in Definition 1.1, therefore, from what we have seen in 3.10, we can construct a chiral $\mathcal{R}$-extension of $\mathcal{R} \otimes \mathcal{R}$ by $\mathcal{R}$ (here $L = \mathcal{R}$ and $L^c = \mathcal{R}^c$)

$$(11) \qquad 0 \to \mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c) \to \mathcal{R} \otimes \mathcal{R} \to 0.$$

Below we will use the above chiral extension to define the chiral algebroid $\Omega^c(\mathcal{R})$.

3.12.   THE CONSTRUCTION OF $\Omega^c(\mathcal{R})$. We can now proceed to the construction of $\Omega^c(\mathcal{R})$. Recall that, because of the Poisson bracket on $\mathcal{R}$, the sheaf $\Omega^1(\mathcal{R})$ acquires a structure of a Lie* algebroid. In fact the action of $\mathcal{R}$ on $\mathcal{R}$ given by the Poisson bracket yields a Lie* $\mathcal{R}$-algebroid structure on $\mathcal{R} \otimes \mathcal{R}$. One checks easily that the kernel of the projection $\mathcal{R} \otimes \mathcal{R} \to \Omega^1(\mathcal{R})$, $z \otimes w \mapsto z\,dw$, is an ideal in $\mathcal{R} \otimes \mathcal{R}$, hence $\Omega^1(\mathcal{R})$ inherits a Lie* algebroid structure.

Recall that we denoted by $\mathcal{Q}^{ch}(\mathcal{R})$ the groupoid of $\mathbb{C}[\hbar]/\hbar^2$-deformations of our chiral-Poisson algebra $\mathcal{R}$, and that we want to understand how to construct the inverse to the functor

$$\mathcal{P}^{ch}(\Omega^1(\mathcal{R})) \to \mathcal{Q}^{ch}(\mathcal{R}),$$

that assigns to a chiral extension $0 \to \mathcal{R} \to \Omega^c(\mathcal{R}) \to \Omega^1(\mathcal{R}) \to 0$, its pullback via the differential $d : \mathcal{R} \to \Omega^1(\mathcal{R})$.

3.13.   The inverse functor will be constructed as follows: for any object in $\mathcal{Q}^{ch}(\mathcal{R})$, i.e. to for any extension $0 \to \mathcal{R} \xrightarrow{\cdot\hbar} \mathcal{R}^c \to \mathcal{R} \to 0$, we will consider the chiral extension

$$0 \to \mathcal{R} \to \mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c) \to \mathcal{R} \otimes \mathcal{R} \to 0$$

described in the previous subsection. We will quotient $\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)$ by some additional relations in order to impose the Leibniz rule on $\mathcal{R} \otimes \mathcal{R}$. These relations will be given, under Kashiwara's equivalence, as the image of a map from $\Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c)$ to $\Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))$. More precisely, we will construct a map from $j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ to $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c))$ such that the composition with the projection $\Delta_!(\mathrm{Ind}_{\mathcal{R}}(\mathcal{R}^c)) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))$ vanishes when restricted to $\mathcal{R}^c \boxtimes \mathcal{R}^c$. Hence it will induce a map $\Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))$. Form the sequence (11) we will therefore obtain a chiral $\mathcal{R}$-extension $\widetilde{\Omega^c(\mathcal{R})}$ of the Lie* algebroid $\Omega^1(\mathcal{R})$

$$0 \to \mathcal{R}' \to \widetilde{\Omega^c(\mathcal{R})} \to \Omega^1(\mathcal{R}) \to 0.$$

---

[2]This correction is due to the fact that, as we saw in 1.2, the equivalence stated in Theorem 1.1 gives a quantization of $1/2\{\,,\,\}$.

We will then check that $\mathcal{R}'$, which a priori is a quotient of $\mathcal{R}$, is in fact $\mathcal{R}$ itself, and that the pull-back via the differential $d : \mathcal{R} \to \Omega^1(\mathcal{R})$ is the original sequence (10), with induced Poisson bracket given by $\{,\}$. This will imply that $\widetilde{\Omega^c(\mathcal{R})}$ is in fact the chiral extension $\Omega^c(\mathcal{R})$ given by Theorem 1.1.

3.14. The map from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ to $\Delta_!(\mathrm{Ind}_\mathcal{R}(\mathcal{R}^c))$ is defined as the sum of the following three maps:

(1) The first map α_1 is given by the composition

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \to j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_\mathcal{R}(\mathcal{R}^c)),$$

where the first map comes from the projection $\mathcal{R}^c \to \mathcal{R}$.

(2) The second map α_2 is obtained from the first one by interchanging the roles of the factors in $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$.

(3) For the third map α_3, note that the chiral bracket $\mu_\hbar$ on $\mathcal{R}_h$ gives rise to a map

$$\cdot\mu_c : j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \to \Delta_!(\mathcal{R}^c)$$

and we compose it with the canonical map $\Delta_!(\mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_\mathcal{R}(\mathcal{R}^c))$.

Now consider the linear combination $\alpha_1 - \alpha_2 - \alpha_3$ as a map from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ to $\Delta_!(\mathrm{Ind}_\mathcal{R}(\mathcal{R}^c))$. If we compose this map with the inclusion $\mathcal{R}^c \boxtimes \mathcal{R}^c \to j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ and the projection onto $\mathrm{Ind}_\mathcal{R}^{ch}(\mathcal{R}^c)$, it is easy to see that the map vanishes. More precisely we have the following:

Lemma 3.6. *The composition*

$$\mathcal{R}^c \boxtimes \mathcal{R}^c \hookrightarrow j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\alpha_1 - \alpha_2 - \alpha_3} \Delta_!(Ind_\mathcal{R}(\mathcal{R}^c)) \twoheadrightarrow \Delta_!(Ind_\mathcal{R}^{ch}(\mathcal{R}^c))$$

vanishes. Thus it defines a map $Leib : \Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \to \Delta_!(Ind_\mathcal{R}^{ch}(\mathcal{R}^c))$.

Proof. Since the action of $\mathcal{R}^c$ on $\mathcal{R}$ is given by the projection $\mathcal{R}^c \to \mathcal{R}$ and the Poisson bracket on $\mathcal{R}$ multiplied by $1/2$, and because of the relation $\sigma \circ \{\,,\,\} \circ \sigma = -\{\,,\,\}$, the maps α_1 and α_2 factor as

$$\mathcal{R}^c \boxtimes \mathcal{R}^c \longrightarrow \Delta_!(\mathrm{Ind}_\mathcal{R}(\mathcal{R}^c)) \longrightarrow \Delta_!(\mathrm{Ind}_\mathcal{R}^{ch}(\mathcal{R}^c)) \ .$$

$$\alpha_1 = \tfrac{1}{2}\{\,,\,\} = -\alpha_2 \searrow \qquad \nearrow$$

$$\Delta_!(\mathcal{R})$$
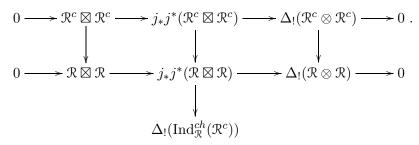
Note that the above wouldn't have been true if we hadn't used the relation in $\mathrm{Ind}_\mathcal{R}(\mathcal{R}^c)$ as well. Moreover the third map, when composed with the projection to $\Delta_!(\mathrm{Ind}_\mathcal{R}^{ch}(\mathcal{R}^c))$ is exactly

$$\mathcal{R}^c \boxtimes \mathcal{R}^c \to \mathcal{R} \boxtimes \mathcal{R} \xrightarrow{\{,\}} \Delta_!(\mathcal{R}) \to \Delta_!(\mathrm{Ind}_\mathcal{R}^{ch}(\mathcal{R}^c)),$$
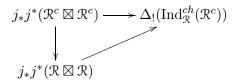
hence the combination $\alpha_1 - \alpha_2 - \alpha_3$ is indeed zero. From the above we therefore get a map $\Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))$.

$\square$

We define $\widetilde{\Omega^c(\mathcal{R})}$ to be the quotient of $\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)$ by the image of the corresponding map from $\mathcal{R}^c \otimes \mathcal{R}^c$ to $\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)$ under the Kashiwara's equivalence.

Remark 3.7. Note that the map $\mathcal{R}^c \otimes \mathcal{R}^c \to \mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)$ indeed factors through $\mathcal{R}^c \otimes \mathcal{R}^c \twoheadrightarrow \mathcal{R} \otimes \mathcal{R}$. To show this it is enough to show that the map $j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))$ factors through $j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \to j_* j^*(\mathcal{R} \boxtimes \mathcal{R})$. If so, then the diagram below would imply that the composition $\mathcal{R} \boxtimes \mathcal{R} \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))$ is zero, and we are done:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R}^c \boxtimes \mathcal{R}^c & \longrightarrow & j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) & \longrightarrow & \Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) & \longrightarrow & 0 \ . \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R} \boxtimes \mathcal{R} & \longrightarrow & j_* j^*(\mathcal{R} \boxtimes \mathcal{R}) & \longrightarrow & \Delta_!(\mathcal{R} \otimes \mathcal{R}) & \longrightarrow & 0 \\
& & & & \downarrow & & & & \\
& & & & \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) & & & &
\end{array}
$$

To show that the map factors as

$$
\begin{array}{ccc}
j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) & \longrightarrow & \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) \\
\downarrow & \nearrow & \\
j_* j^*(\mathcal{R} \boxtimes \mathcal{R}) & &
\end{array}
$$

we need to show that the composition of the map $\alpha_1 - \alpha_2 - \alpha_3$ with the two embeddings $j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}) \hookrightarrow j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ and $j_* j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \hookrightarrow j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ is zero. We'll do only one of them (the second one can be done similarly). For the first embedding the map α_2 is zero (since we are projecting the second $\mathcal{R}^c$ onto $\mathcal{R}$) whereas the first map (because of the relations in $\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)$) is equal to minus the composition

$$
j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R}) \to j_* j^*(\mathcal{R} \boxtimes \mathcal{R}) \xrightarrow{\mu} \Delta_!(\mathcal{R}) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))
$$

which is exactly the third map when restricted to $j_* j^*(\mathcal{R}^c \boxtimes \mathcal{R})$.

3.15. Recall that we defined $\widetilde{\Omega^c(\mathcal{R})}$ to be the quotient of $\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)$ by the image of the map *Leib* obtained using the combination α_1-α_2-α_3. By construction we have a short exact sequence

$$(12) \qquad\qquad 0 \to \mathcal{R}' \to \widetilde{\Omega^c(\mathcal{R})} \to \Omega^1(\mathcal{R}) \to 0,$$

where $\mathcal{R}'$ is a certain quotient of $\mathcal{R}$. In the rest of this section we will show that the above extension is in fact isomorphic to the extension of $\Omega^1(\mathcal{R})$ given in Theorem 1.1. This is equivalent to the following:

Proposition 3.8. *Consider the extension of $\Omega^1(\mathcal{R})$ given by (12). Then we have*

(1) $\mathcal{R}' = \mathcal{R}$.

(2) *The pull-back of (12) via the differential $d : \mathcal{R} \to \Omega^1(\mathcal{R})$ is the original sequence (10).*

Proof. To show that $\mathcal{R}' = \mathcal{R}$, consider the chiral extension given by the equivalence of Theorem 1.1. This is an extension of $\Omega^1(\mathcal{R})$ such that the pull back via the differential $\mathcal{R} \to \Omega^1(\mathcal{R})$ is the sequence (10). that is, we have the following diagram

(13)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R} & \overset{i}{\longrightarrow} & \Omega^c(\mathcal{R}) & \longrightarrow & \Omega^1(\mathcal{R}) & \longrightarrow & 0, \\
& & \uparrow & & \uparrow{\scriptstyle d^c} & & \uparrow{\scriptstyle d} & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{R}^c & \overset{\pi}{\longrightarrow} & \mathcal{R} & \longrightarrow & 0
\end{array}
$$

with d^c a derivation, i.e. as maps from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$ to $\Delta_!(\Omega^c(\mathcal{R}))$, we have $d^c(\mu_c) = \mu_{\mathcal{R},\Omega^c(\mathcal{R})}(\pi, d^c) - \sigma \circ \mu_{\mathcal{R},\Omega^c(\mathcal{R})} \circ \sigma(d^c, \pi)$, where μ_c is the chiral product on $\mathcal{R}^c$ and $\mu_{\mathcal{R},\Omega^c(\mathcal{R})}$ is the chiral action of $\mathcal{R}$ on $\Omega^c(\mathcal{R})$. We claim that there is a map of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta_!(\mathcal{R}) & \longrightarrow & \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)) & \longrightarrow & \Delta_!(\mathcal{R} \otimes \mathcal{R}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta_!(\mathcal{R}) & \longrightarrow & \Delta_!(\Omega^c(\mathcal{R})) & \longrightarrow & \Delta_!(\Omega^1(\mathcal{R})) & \longrightarrow & 0
\end{array}
$$

that factors through $0 \to \Delta_!(\mathcal{R}') \to \Delta_!(\widetilde{\Omega^c(\mathcal{R})}) \to \Delta_!(\Omega^1(\mathcal{R})) \to 0$, and moreover induces an isomorphism from $\Omega^1(\mathcal{R})$ to $\Omega^1(\mathcal{R})$. This would imply that $\mathcal{R}'$, which a priori is a quotient of $\mathcal{R}$, is in fact $\mathcal{R}$ itself. Furthermore, the fact that it is an isomorphism on $\Omega^1(\mathcal{R})$, would also imply that $\widetilde{\Omega^c(\mathcal{R})} \simeq \Omega^c(\mathcal{R})$, hence the pull-back via $d : \mathcal{R} \to \Omega^1(\mathcal{R})$ would indeed be the original sequence $0 \to \mathcal{R} \to \mathcal{R}^c \to \mathcal{R} \to 0$.

To prove the claim, consider the map $d^c : \mathcal{R}^c \to \Omega^c(\mathcal{R})$ given by (13). Using the chiral $\mathcal{R}$-module structure $\mu_{\mathcal{R},\Omega^c(\mathcal{R})}$ on $\Omega^c(\mathcal{R})$, we can consider the composition

$$
j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \xrightarrow{i \boxtimes d^c} j_*j^*(\mathcal{R} \boxtimes \Omega^c(\mathcal{R})) \xrightarrow{\mu_{\mathcal{R},\Omega^c(\mathcal{R})}} \Delta_!(\Omega^c(\mathcal{R})).
$$

The above composition can be extended to a map from $\Delta_!(\mathcal{R}) \oplus j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \to \Delta_!(\Omega^c(\mathcal{R}))$, by setting the map to be $\Delta_!(i)$ on $\Delta_!(\mathcal{R})$. It is straightforward to check that this map factors through a map D^c

$$
D^c : \Delta_!(\mathrm{Ind}^{ch}_{\mathcal{R}}(\mathcal{R}^c)) \to \Delta_!(\Omega^c(\mathcal{R})).
$$

Note that, by construction, the resulting map $\overline{D}^c : \mathcal{R} \otimes \mathcal{R} \to \Omega^1(\mathcal{R})$ is the one given by $z \otimes w \mapsto z\,dw$, for z and w in $\mathcal{R}$, and that the kernel of this

map is just the ideal defining the Leibniz rule.

To show that D^c factors through $0 \to \Delta_!(\mathcal{R}') \to \Delta_!(\widetilde{\Omega^c(\mathcal{R})}) \to \Delta_!(\Omega^1(\mathcal{R})) \to$ 0, we need to show that the composition of D^c with the map

$$Leib : \Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c))$$

given in 3.13. vanishes. Hence we are left with checking that the composition

$$\Delta_!(\mathcal{R}^c \otimes \mathcal{R}^c) \xrightarrow{Leib} \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) \xrightarrow{\Delta_!(D^c)} \Delta_!(\Omega^c(\mathcal{R}))$$

is zero. For this, recall that the map $Leib$ was constructed using the linear combination $\alpha_1 - \alpha_2 - \alpha_3$ of three maps α_1, α_2 and α_3 from $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c)$. By looking at the map

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\alpha_1 - \alpha_2 - \alpha_3} \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) \xrightarrow{\Delta_!(D^c)} \Delta_!(\Omega^c(\mathcal{R})),$$

we see that the condition on d^c being a derivation, implies that the above composition vanishes. Indeed $\Delta_!(D^c) \circ \alpha_1$ is given by

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\pi \boxtimes id} j_*j^*(\mathcal{R} \boxtimes \mathcal{R}^c) \xrightarrow{id \boxtimes d^c} {}_{J_*}j^*(\mathcal{R} \boxtimes \Omega^c(\mathcal{R})) \xrightarrow{\mu_{\mathcal{R}, \Omega^c(\mathcal{R})}} \Delta_!(\Omega^c(\mathcal{R})).$$

The map $\Delta_!(D^c) \circ \alpha_2$ is given by the above by applying the transposition of variables σ, whereas the third map

$$j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\mu_c} \Delta_!(\mathcal{R}^c) \to \Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) \xrightarrow{D^c} \Delta_!(\Omega^c(\mathcal{R}))$$

is equal to $j_*j^*(\mathcal{R}^c \boxtimes \mathcal{R}^c) \xrightarrow{\mu_c} \Delta_!(\mathcal{R}^c) \xrightarrow{\Delta_!(d^c)} \Delta_!(\Omega^c(\mathcal{R}))$. Therefore the above maps coincide with the terms in the relation $d^c(\mu_c) = \mu_{\mathcal{R}, \Omega^c(\mathcal{R})}(\pi, d^c) - \sigma \circ \mu_{\mathcal{R}, \Omega^c(\mathcal{R})} \circ \sigma(d^c, \pi)$, and hence $\Delta_!(D^c) \circ Leib$ in zero. Note that the resulting map

$$\Delta_!(\mathrm{Ind}_{\mathcal{R}}^{ch}(\mathcal{R}^c)) \xrightarrow{\pi} \Delta_!(\mathcal{R} \otimes \mathcal{R}) \xrightarrow{\Delta_!(\overline{D^c})} \Delta_!(\Omega^1(\mathcal{R}))$$

induces an isomorphism

$$\Delta_!(\Omega^1(\mathcal{R})) \simeq \Delta_!(\mathcal{R} \otimes \mathcal{R})/\mathrm{Im}(\pi \circ Leib) \xrightarrow{\sim} \Delta_!(\Omega^1(\mathcal{R})).$$

This conclude the proof of the proposition.   □

4. Construction of the map F

4.1.   Recall that, by definition, Theorem 2 amounts to the construction of a map of Lie* algebras $F : \Omega^c(\mathfrak{Z}_{crit}) \to \mathcal{B}^0$ compatible with the $\mathfrak{Z}_{crit}$ structure on both sides and such that

  (1) F restricts to the embedding l (given in Remark 2.4) on $\mathfrak{Z}_{crit}$.

  (2) The following diagram commutes:

$$\begin{array}{ccc} \Omega^c(\mathfrak{Z}_{crit})/\mathfrak{Z}_{crit} & \xrightarrow{\quad F \quad} & (\mathcal{B}^0)_1/\mathfrak{Z}_{crit}. \\ & \nwarrow {\scriptstyle \simeq} \quad\quad {\scriptstyle \simeq} \nearrow & \\ & \Omega^1(\mathfrak{Z}_{crit}) & \end{array}$$

4.2. Since $\Delta_!(\Omega^c(\mathfrak{Z}_{crit}))$ was constructed as a quotient of $\Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}_{crit}^c))$ and since, by definition,

$$\Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}_{crit}^c)) = \Delta_!(\mathfrak{Z}_{crit}) \oplus j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit}^c)/\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit}^c,$$

to construct any map F from $\Omega^c(\mathfrak{Z}_{crit})$ to $\mathcal{B}^0$ we can proceed as follows:

- first we construct a map $f : \mathfrak{Z}_{crit}^c \to \mathcal{B}^0$.
- Using the chiral bracket μ' on $\mathcal{B}^0$ we consider the composition

$$j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit}^c) \xrightarrow{l \boxtimes f} j_*j^*(\mathcal{B}^0 \boxtimes \mathcal{B}^0) \xrightarrow{\mu'} \Delta_!(\mathcal{B}^0).$$

  This composition yields a map

$$\hat{F} : \Delta_!(\mathfrak{Z}_{crit}) \oplus j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit}^c) \to \Delta_!(\mathcal{B}^0),$$

  by sending $\Delta_!(\mathfrak{Z}_{crit})$ to $\mathcal{B}^0$ via $\Delta_!(l)$.
- We check that the above map factors through a map

$$\widetilde{F} : \Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}_{crit}^c)) \to \Delta_!(\mathcal{B}^0).$$

- We check that in fact if factors through $\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}_{crit}^c)) \to \Delta_!(\mathcal{B}^0)$.
- We verify that the relations defining $\Delta_!(\Omega^c(\mathfrak{Z}_{crit}))$ as a quotient of $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}_{crit}^c))$ are satisfied, i.e. that $\overline{F}$ gives the desired map F from $\Omega(\mathfrak{Z}_{crit})^c$ to $\mathcal{B}^0$ under the Kashiwara equivalence.

Remark 4.1. Note that any map F constructed as before, automatically satisfies the first condition in 4.1, hence to prove Theorem 2, once the map f is defined, we only have to verify that condition (2) in 4.1 is satisfied, i.e. that the diagram above commutes.

4.3. DEFINITION OF THE MAP f. We will now define the map $f : \mathfrak{Z}_{crit}^c \to \mathcal{B}^0$ and hence, according to the first two points in 4.2, the map $\hat{F} : \Delta_!(\mathfrak{Z}_{crit}) \oplus j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit}^c) \to \Delta_!(\mathcal{B}^0)$. In 4.5, assuming that it factors through a map $F : \Omega^c(\mathfrak{Z}_{crit}) \to \mathcal{B}^0$, we will then show that it satisfies the second condition in 4.1. This will conclude the proof of Theorem 2. The poof that it factors through $\Omega^c(\mathfrak{Z}_{crit})$ (which amounts to the proof of the remaining last three points in 4.2) will be postponed until 4.6.

4.4. To define the map $f : \mathfrak{Z}_{crit}^c \to \mathcal{B}^0$ we will use the following three facts:

(1) There exist two embeddings

$$\mathfrak{Z}_{crit} \xrightarrow{l} \mathcal{B}^0 \xleftarrow{r} \mathfrak{Z}_{crit}$$

constructed by applying the functor Ψ to the two embeddings in (4). In fact, by doing it, we obtain two maps

$$\mathcal{W}_\hbar \xrightarrow{l_\hbar} (\Psi \boxtimes \Psi)(\mathcal{D}_\hbar) \xleftarrow{r_\hbar} \mathcal{W}_{-\hbar}$$

such that $l := l_0 = r_0 \circ \eta =: r \circ \eta$, where we are denoting by $l_\hbar$ and $r_\hbar$ the maps $\Psi(l_\hbar)$ and $\Psi(r_\hbar)$ respectively. The two embedding of $\mathfrak{Z}_{crit}$ correspond to the above maps when $\hbar = 0$.

(2) There is a well defined map

$$e : \mathcal{W}_\hbar \to \mathcal{W}_{-\hbar}.$$

In fact, since $\mathcal{W}_\hbar = \Psi(\mathcal{A}_\hbar)$, and since $\mathcal{A}_{-\hbar}$ is isomorphic to $\mathcal{A}_\hbar$ as vector space with the action of $\mathbb{C}[\hbar]$ modified to $\hbar \cdot a = -\hbar a$, $a \in \mathcal{A}_{-\hbar}$, we can consider the map $\mathcal{W}_\hbar \to \mathcal{W}_{-\hbar}$ that simply sends $\hbar$ to $-\hbar$.

(3) The involution $\eta : \mathfrak{Z}_{crit} \to \mathfrak{Z}_{crit}$ can be extended to a map $\eta : \mathcal{W}_\hbar \to \mathcal{W}_\hbar$ by setting $\eta(h) = h$.

We define f in the following way: for every $z_\hbar \in \mathfrak{Z}_{crit}^c = \mathcal{W}_\hbar / \hbar^2 \mathcal{W}_\hbar$ we set

$$f(z_\hbar) = \frac{1}{2} \frac{l_\hbar(z_\hbar) - r_\hbar(\eta(e(z_\hbar)))}{\hbar} \quad (\text{mod } \hbar).$$

This is a well defined element in $\mathcal{B}^0$ because $l_0 = r_0 \circ \tau$, i.e. the numerator vanishes mod $\hbar$.

Assuming the proposition below, we will now show that the resulting F satisfies condition (2) of 4.1, which, according to Remark 4.1, concludes the proof of Theorem 2. Proposition 4.2 will be proved later in 4.6.

Proposition 4.2. *The map*

$$\hat{F} : \Delta_!(\mathfrak{Z}_{crit}) \oplus j_* j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit}^c) \to \Delta_!(\mathcal{B}^0),$$

obtained by using $f : \mathfrak{Z}_{crit}^c \to \mathcal{B}^0$ *from above, factors through a map* $F : \Delta_!(\Omega^c(\mathfrak{Z}_{crit})) \to \Delta_!(\mathcal{B}^0)$.

4.5. END OF THE PROOF OF THEOREM 2.

Proof. We are now ready to finish the proof of Theorem 2, which, according to Remark 4.1, amounts to check that

$$\Omega^c(\mathfrak{Z}_{crit})/\mathfrak{Z}_{crit} \xrightarrow{\quad F \quad} (\mathcal{B}^0)_1/\mathfrak{Z}_{crit}$$

$$\Omega^1(\mathfrak{Z}_{crit})$$

commutes. In order to do so, we will show that it commutes when composed with the map $d : \mathfrak{Z}_{crit} \to \Omega^1(\mathfrak{Z}_{crit})$. By looking at the composition

$$\mathfrak{Z}_{crit} \xrightarrow{d} \Omega^1(\mathfrak{Z}_{crit}) \to \Omega^c(\mathfrak{Z}_{crit})/\mathfrak{Z}_{crit} \xrightarrow{F} (\mathcal{B}^0)_1/\mathfrak{Z}_{crit},$$

we see that, for $z \in \mathfrak{Z}_{crit}$, the resulting map is

(14) $$z \mapsto \frac{1}{2} \frac{l_\hbar(z_\hbar) - r_\hbar(\eta(e(z_\hbar)))}{\hbar} \quad (\text{mod } \hbar),$$

where $z_\hbar$ is any lifting of z to $\mathfrak{Z}_{crit}^c$. Note that this map is well defined only after taking the quotient of $\mathcal{B}^0$ by $\mathfrak{Z}_{crit}$.

For the other composition, we first need to recall how the isomorphism $\Omega^1(\mathfrak{Z}_{crit}) \xrightarrow{\sim} (\mathcal{B}^0)_1/\mathfrak{Z}_{crit}$ was constructed. Recall from 2.3 that the filtration

on $\mathcal{B}^0$ is the one induced (under $\Psi \boxtimes \Psi$) from the isomorphism G given in Theorem 2.2. Therefore the isomorphism above is the one corresponding to the composition

$$(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \underset{\mathfrak{Z}_{crit}}{\otimes} \Omega^1(\mathfrak{Z}_{crit}) \xrightarrow{\sim} U(\mathcal{A}^{ren,\tau})_1/(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}) \xrightarrow{G}$$

$$\to \mathcal{D}^0_{crit}/l(\mathcal{A}_{crit}) + r(\mathcal{A}_{crit})$$

under $(\Psi \boxtimes \Psi)$ (here, for simplicity, we are denoting the chiral envelope $U((\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit}), \mathcal{A}^{ren,\tau})$ by $U(\mathcal{A}^{ren,\tau})$). If we consider the inclusion of $\Omega^1(\mathfrak{Z}_{crit})$ followed by the first arrow from above, it is clear that the image in $U(\mathcal{A}^{ren,\tau})/(\mathcal{A}_{crit} \underset{\mathfrak{Z}_{crit}}{\otimes} \mathcal{A}_{crit})$ is $G[[t]] \times G[[t]]$ invariant. In particular it means that the image of $\Omega^1(\mathfrak{Z}_{crit})$ maps to $(\Psi \boxtimes \Psi)(U(\mathcal{A}^{ren,\tau}))/\mathfrak{Z}_{crit}$. Now, by looking at the definition of the map G (see [FG] 5.5.), we see that the the map

$$\mathfrak{Z}_{crit} \xrightarrow{d} \Omega^1(\mathfrak{Z}_{crit}) \to (\Psi \boxtimes \Psi)(U(\mathcal{A}^{ren,\tau}))_1/\mathfrak{Z}_{crit} \xrightarrow{(\Psi \boxtimes \Psi)(G)} (\mathcal{B}^0)_1/\mathfrak{Z}_{crit},$$

is indeed given by (14). This completes the proof of Theorem 1. $\square$

We will now give the proof of Proposition 4.2, which will occupy the rest of the article.

4.6. Proof of Proposition 4.2.

Proof. Recall that the proof of Proposition 4.2 consists in showing the following:

(1) the map $\hat{F} : \Delta_!(\mathfrak{Z}_{crit}) \oplus j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \to \Delta_!(\mathcal{B}^0)$ factors through a map

$$\widetilde{F} : \Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \to \Delta_!(\mathcal{B}^0).$$

(2) the map $\widetilde{F}$ factors through $\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \to \Delta_!(\mathcal{B}^0)$.

(3) The relations defining $\Delta_!(\Omega^c(\mathfrak{Z}_{crit}))$ as a quotient of $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$ are satisfied, i.e. $\overline{F}$ gives the desired map F from $\Omega(\mathfrak{Z}_{crit})^c$ to $\mathcal{B}^0$ under the Kashiwara equivalence.

For this we will need the following Lemma.

Lemma 4.3. *The composition*

$$\mathcal{W}_\hbar \boxtimes \mathcal{W}_{-\hbar} \xrightarrow{l_\hbar \boxtimes r_\hbar} (\Psi \boxtimes \Psi)(\mathcal{D}_\hbar) \boxtimes (\Psi \boxtimes \Psi)(\mathcal{D}_\hbar) \xrightarrow{\mu'} \Delta_!((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar))$$

is zero.

Proof. In [FG] Lemma 5.2 it is shown that the composition

$$\mathcal{A}_\hbar \boxtimes \mathcal{A}_{-\hbar} \xrightarrow{l_\hbar \boxtimes r_\hbar} \mathcal{D}_\hbar \boxtimes \mathcal{D}_\hbar \xrightarrow{\mu'} \Delta_!(\mathcal{D}_\hbar)$$

is zero. In other words the two embeddings centralize each other. The Lemma then, immediately follows by applying the functor $(\Psi \boxtimes \Psi)$.

$\square$

4.7.   PROOF OF (1). To prove that $\hat{F}$ factors through

$$\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \to \Delta_!(\mathcal{B}^0)$$

we use Lemma 4.3. Recall that we defined f from $\mathfrak{Z}^c_{crit}$ to $\mathcal{B}^0$ to be

$$f(z_\hbar) = \frac{1}{2}\frac{l_\hbar(z_\hbar) - r_\hbar(\eta(e(z_\hbar)))}{\hbar} \quad (\mathrm{mod}\ \hbar).$$

Because of the above Lemma, it is clear that, when we consider the inclusion $\mathfrak{Z}_{crit}\boxtimes\mathfrak{Z}^c_{crit} \hookrightarrow j_*j^*(\mathfrak{Z}_{crit}\boxtimes\mathfrak{Z}^c_{crit})$ and the composition with the map to $\Delta_!(\mathcal{B}^0)$, the resulting map factors as:

$$
\begin{array}{ccccc}
\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit} & \longrightarrow & j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit}) & \longrightarrow & \Delta_!(\mathcal{B}^0) \ , \\
\downarrow & & & \nearrow \scriptstyle{\Delta_!(l)} & \\
\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit} & \xrightarrow{\frac{1}{2}\{,\}} & \Delta_!(\mathfrak{Z}_{crit}) & &
\end{array}
$$

which implies that the map factors through a map $\widetilde{F} : \Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \to \Delta_!(\mathcal{B}^0)$.

Remark 4.4. Note that when we restrict the map $f : \mathfrak{Z}^c_{crit} \to \mathcal{B}^0$ to $\mathfrak{Z}_{crit} \xrightarrow{\cdot\hbar} \mathfrak{Z}^c_{crit}$, because of the flip from $\hbar$ to $-\hbar$ in the definition of e, we simply obtain the inclusion $\mathfrak{Z}_{crit} \xrightarrow{l} \mathcal{B}^0$.

4.8.   PROOF OF (2). Now we want to check that the relations defining $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$ as a quotient of $\Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$ are satisfied, i.e. that $\widetilde{F}$ factors through a map $\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \to \Delta_!(\mathcal{B}^0)$.
First of all, recall that to pass from $\Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$ to $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$ we took the quotient by the image of the difference of two maps from $\Delta_!(\mathfrak{Z}_{crit} \otimes \mathfrak{Z}_{crit})$ to $\Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$. The first map was given by

$$(15) \qquad \Delta_!(\mathfrak{Z}_{crit} \otimes \mathfrak{Z}_{crit}) \xrightarrow{m} \Delta_!(\mathfrak{Z}_{crit}) \to \Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})),$$

while the second map was induced by the composition

$$j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit}) \xrightarrow{id\boxtimes\cdot h} j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \twoheadrightarrow \Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})),$$

which vanishes on $\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit} \hookrightarrow j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}_{crit})$. When we compose the map (15) with $\widetilde{F}$, we get $l \circ m = \mu' \circ (l \boxtimes l)$. However when we compose the second map with

$$j_*j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \xrightarrow{l\boxtimes f} j_*j^*(\mathcal{B}^0 \boxtimes \mathcal{B}^0) \xrightarrow{\mu'} \Delta_!(\mathcal{B}^0),$$

because of Remark 4.4, we see that this map corresponds to $\mu' \circ (l \boxtimes l)$ hence the difference of the images goes to zero under $\widetilde{F}$.

4.9. PROOF OF (3). Now we are left with checking that $\widetilde{F}$ factors through

$$\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \to \Delta_!(\Omega^c(\mathfrak{Z}_{crit})).$$

This will occupy the rest of the article. Recall that $\Delta_!(\Omega^c(\mathcal{R}))$ was given as a quotient of $\Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$ by the map *Leib*. The Leibniz relation was given as the image of a map

$$\Delta_!(\mathfrak{Z}^c_{crit} \otimes \mathfrak{Z}^c_{crit}) \to \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit}))$$

and this map was the sum of three maps, α_1, α_2 and α_3, from $j_* j^*(\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit})$ which vanished on $\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit}$. Hence we want to check that the composition

$$\Delta_!(\mathfrak{Z}^c_{crit} \otimes \mathfrak{Z}^c_{crit}) \xrightarrow{Leib} \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \xrightarrow{\overline{F}} \Delta_!(\mathcal{B}^0)$$

vanishes. Instead of considering the map from $\Delta_!(\mathfrak{Z}^c_{crit} \otimes \mathfrak{Z}^c_{crit})$ we can consider the three maps

$$(16) \qquad j_* j^*(\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \xrightarrow[\alpha_3]{\overset{\alpha_1}{\underset{\alpha_2}{\Longrightarrow}}} \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \xrightarrow{\overline{F}} \Delta_!(\mathcal{B}^0),$$

and show that the composition $\overline{F} \circ (\alpha_1 - \alpha_2 - \alpha_3)$ is zero. Recall that the first map, α_1, was given by projecting onto $j_* j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit})$, i.e.

$$j_* j^*(\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \xrightarrow{\hspace{2cm}} j_* j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \xrightarrow[\beta]{\overset{\alpha_1}{\hspace{2cm}}} \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \ ,$$

where β denotes the second component of the projection

$$\Delta_!(\mathfrak{Z}_{crit}) \oplus j_* j^*(\mathfrak{Z}_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \twoheadrightarrow \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})).$$

The second map, α_2, was given by $\sigma \circ \alpha_1 \circ \sigma$, and the third map α_3 was given by the composition

$$j_* j^*(\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \xrightarrow{\mu_c} \Delta_!(\mathfrak{Z}^c_{crit}) \xrightarrow{\bar{i}} \Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \twoheadrightarrow$$
$$\twoheadrightarrow \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})).$$

When we compose α_3 with the map $\overline{F} : \Delta_!(\mathrm{Ind}^{ch}_{\mathfrak{Z}_{crit}}(\mathfrak{Z}^c_{crit})) \to \Delta_!(\mathcal{B}^0)$, it is easy to see that the unit axiom implies that the composition is equal to

$$(17) \qquad j_* j^*(\mathfrak{Z}^c_{crit} \boxtimes \mathfrak{Z}^c_{crit}) \xrightarrow{\mu_c} \Delta_!(\mathfrak{Z}^c_{crit}) \xrightarrow{\Delta_!(f)} \Delta_!(\mathcal{B}^0).$$

Now consider the chiral algebra $(\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)$ and denote by $\mu'_\hbar$ its chiral operation. Consider the map

$$f_\hbar : \mathcal{W}_\hbar \to (\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)$$

$$f_\hbar(z_\hbar) = \frac{1}{2} \frac{l_\hbar(z_\hbar) - r_\hbar(\eta(e(z_\hbar)))}{\hbar} \in (\Psi \boxtimes \Psi)(\mathcal{D}_{g,h}),$$

(i.e. we are not taking this element $\pmod{\hbar}$). It is clear that the three maps α''_1, α''_2 and α''_3 given by

$$j_* j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{l_\hbar \boxtimes f_\hbar} j_* j^*((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar) \boxtimes (\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \xrightarrow{\mu'_\hbar}$$

(α_1'') $\qquad\qquad\qquad\qquad\xrightarrow{\mu_\hbar'} \Delta_!((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \to \Delta_!(\mathcal{B}^0),$

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{(r_\hbar \circ \eta \circ e \boxtimes f_\hbar) \circ \sigma} j_*j^*((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar) \boxtimes (\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \xrightarrow{\sigma \circ \mu_\hbar'}$$

(α_2'') $\qquad\qquad\qquad\qquad\xrightarrow{\sigma \circ \mu_\hbar'} \Delta_!((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \to \Delta_!(\mathcal{B}^0),$

(α_3'') $\quad j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{\mu_\hbar} \Delta_!(\mathcal{W}_\hbar) \xrightarrow{\Delta_!(f_\hbar)} \Delta_!((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \to \Delta_!(\mathcal{B}^0),$

respectively, vanish on $j_*j^*(\hbar^2(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar)) \hookrightarrow j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar)$, in particular they define well defined maps from $j_*j^*(\mathfrak{Z}_{crit}^c \boxtimes \mathfrak{Z}_{crit}^c)$ to $\Delta_!(\mathcal{B}^0)$. Moreover the resulting maps coincide with α_1, α_2 and α_3 composed with $\overline{F}$. In fact, the first and the last coincide by definition. For the second one, simply note that, modulo $\hbar$, the map $r_\hbar \circ \eta \circ e$ equals l.

By the above, to show that the combination of the three maps given in (16) is zero, it is enough to check that the combination $\alpha_1'' - \alpha_2'' - \alpha_3''$ of the above three maps vanishes.
Let us denote by α_1', α_2' and α_3' the maps from $j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar)$ to $\Delta_!((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar))$ corresponding to α_1'', α_2'' and α_3'' respectively (i.e. before taking the maps $\pmod \hbar$). We will show that the combination $\alpha_1' - \alpha_2' - \alpha_3'$ is already zero.
Because $(\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)$ is h-torsion free, it is enough to show that the three maps agree after multiplication by h. But now note that each of the maps

$$h\alpha_1', \ h\alpha_2', \ h\alpha_3' \in \mathrm{Hom}(j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar), \Delta_!((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar))),$$

is the sum of two terms, and the sum of the resulting six maps is zero. Indeed $h\alpha_1'$ equals the sum of the following two maps:

$$(18) \qquad j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{l_\hbar \boxtimes l_\hbar} j_*j^*((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \xrightarrow{\mu_\hbar'} \Delta_!(\mathcal{D}_\hbar),$$

$$(19) \qquad j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{l_\hbar \boxtimes r_\hbar \circ \eta \circ e} j_*j^*((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \xrightarrow{\mu_\hbar'} \Delta_!(\mathcal{D}_\hbar).$$

On the other hand, the map $h\alpha_3'$ is given by the sum of the following

$$(20) \qquad j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{\mu_\hbar} \Delta_!(\mathcal{W}_\hbar) \xrightarrow{\Delta_!(l_\hbar)} \Delta_!(\mathcal{D}_\hbar),$$

$$(21) \qquad j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{\eta \circ \mu_\hbar \circ (e \boxtimes e)} \delta_!(\mathcal{W}_{-\hbar}) \xrightarrow{\Delta_!(r_\hbar)} \Delta_!(\mathcal{D}_\hbar).$$

It is clear that the map (18) equals minus the map (20). Similarly, the relation $\mu_\hbar' = -\sigma \circ \mu_\hbar' \circ \sigma$ guarantees that the two maps summing up to $h\alpha_2'$, given by

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{(r_\hbar \circ \eta \circ e \boxtimes l_\hbar) \circ \sigma} j_*j^*((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \xrightarrow{\sigma \circ \mu_\hbar'} \Delta_!(\mathcal{D}_\hbar),$$

and

$$j_*j^*(\mathcal{W}_\hbar \boxtimes \mathcal{W}_\hbar) \xrightarrow{(r_\hbar \circ \eta \circ e \boxtimes r_\hbar \circ \eta \circ e) \circ \sigma} j_*j^*((\Psi \boxtimes \Psi)(\mathcal{D}_\hbar)) \xrightarrow{\sigma \circ \mu'_\hbar} \Delta_!(\mathcal{D}_\hbar),$$

cancel with the remaining maps (19) and (21) respectively.

Hence the composition $\alpha_1 - \alpha_2 - \alpha_3$ as a map from $j_*j^*(\mathfrak{Z}_{crit}^c \boxtimes \mathfrak{Z}_{crit}^c)$ to $\Delta_!(\mathcal{B}^0)$ is zero, i.e. the map $\overline{F} : \Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}^{ch}(\mathfrak{Z}_{crit}^c)) \to \Delta_!(\mathcal{B}^0)$ factors as

$$\Delta_!(\mathrm{Ind}_{\mathfrak{Z}_{crit}}^{ch}(\mathfrak{Z}_{crit}^c)) \xrightarrow{\overline{F}} \Delta_!(\mathcal{B}^0) \ .$$

$$\downarrow \qquad \nearrow F$$

$$\Delta_!(\Omega(\mathfrak{Z}_{crit})^c)$$

By Kashiwara we obtain the desired map $F : \Omega(\mathfrak{Z}_{crit})^c \to \mathcal{B}^0$, and this concludes the proof of Theorem 2.

$\square$

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